Rectilinear and Polyomino Knot Projections (See Addendum)

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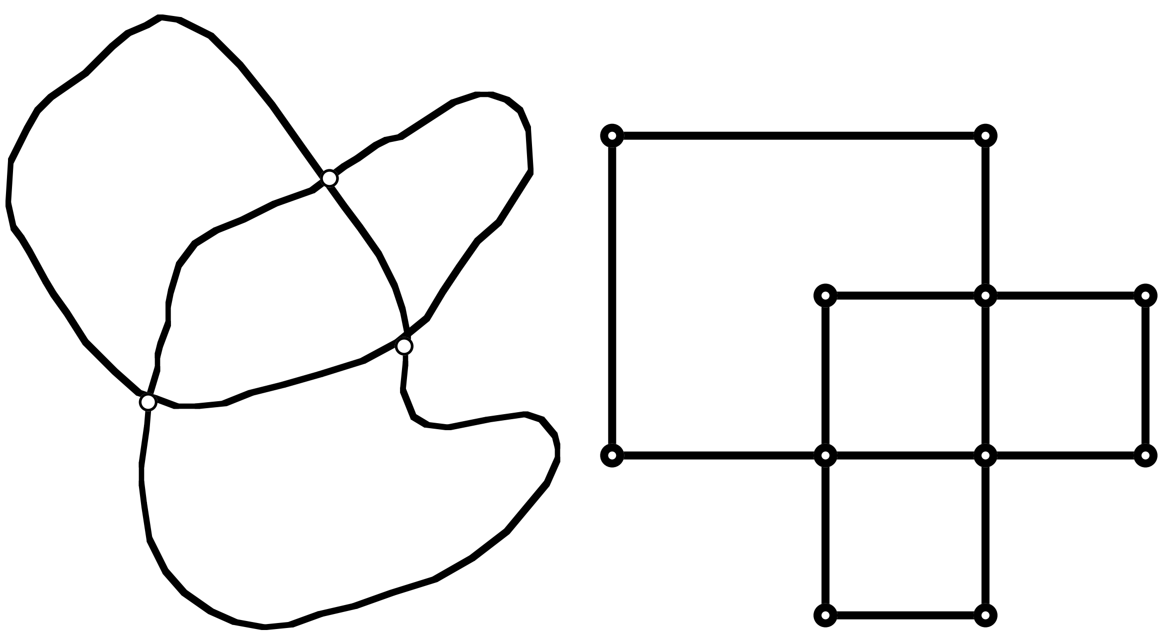
Setting the stage

The purpose of this note is to study knots by looking at them from different combinatorial and geometric perspectives. A knot (piece of string with its ends tied together) in 3-dimensional space can be projected into a plane so that all of the vertices of the resulting plane graph (see Figure 1(a)) are 4-valent (degree 4).



(a) (b)

Figure 1 ((a) A freehand drawing of the trefoil knot in the plane, the graph G(K). (b) Drawing of the graph of the trefoil knot G*(K) using rectilinear polygon faces.)

Usually drawings of knots in the plane are shown in such a way that under-over information about the strands of the knot are coded, but here I will generally disregard under-over information. By adding vertices of valence 2 it is always possible to draw the projection of a knot K so that each region of the plane graph G(K) associated with K can be represented by a simple polygon with an even number of sides. This second special plane graph associated with knot K will be denoted G*(K). Figure 1(b) shows G*(trefoil) associated with the trefoil knot drawn in the plane (Figure 1(a)) with "extra" 2-valent vertices, so that all the edges meet at angles of 0, 90, 180, or 270 degrees. Also, all of the faces including the infinite face have an even number of sides. It is convenient to refer to the shape of the faces in Figure 1(b) as rectilinear (orthogonal) polygons. I will use p_i to denote the number of bounded faces in G(K) with i sides and p^*_i for the number of bounded faces in G*(K). In Figure 1(b) there are three bounded faces with 4 sides, one bounded face with 6 sides and an unbounded face with 10 sides. I will use n to denote the number of sides of the unbounded face (in both G(k) and G*(K). For Figure 1(b) we have n = 10 and $p^*_4 = 3$ and $p^*_6 = 1$ while the equivalent numbers for G(K) (Figure 1a) are n = 3 and $p_2 = 2$ and $p_3 = 1$. In many discussions of plane graphs (graphs drawn in the plane so edges meet only at vertices) it is customary to treat the unbounded face as just another face. From this point of view we have (Figure 1(b) $p^*_4 = 3$, $p^*_6 = 1$ and $p^*_{10} = 1$. However, here I will want to use n as a "parameter" since my goal is, in part, to emphasize the difference between thinking of certain diagrams drawn in the plane from different points of view: plane graph, non-convex polygon, rectilinear polygon (where self-intersections are allowed), and polyominoes (a collection of squares drawn in the plane which are attached edge to edge). Thus, one can regard Figure 1(b) as a self-intersecting polygon drawn in the plane where the "self-intersections" of the sides of the polygon occur at the 4-valent vertices. From this point of view, the number of sides of the polygon would be 8 and the vertices of this polygon are the eight 2-valent vertices. I will use m to denote the number of sides of polygons such as the diagram in Figure 1(b) regarded as a self-intersecting polygon.

Different insight windows into knot projections

Drawings of the kind in Figure 1(b) are always possible for any knot and more generally for any link (union of knots). Again, note that in these diagrams no attempt is made to code the under and over information of the knot at a crossing. The fact that this trefoil knot has a minimum of 3 crossings (in any drawing in the plane where only two strands are allowed to meet at a crossing) is "coded" by the presence of three vertices which are 4-valent. Note that because the under and over information of the drawings is not used, inequivalent knots can have the same graph. Thus, the "unknot" (a knot that can be distorted so it projects to a circle) can have Figure 1(a) as a projection drawing. The number of 2-valent and 4-valent vertices in rectilinear drawings will be denoted by v_2 and v_4. In Figure 1(b) we have $v_2 = 8$ and $v_4 = 3$. You can verify that Euler's Formula, Vertices + Faces - Edges = 2, true for any plane connected graph, holds since we have that in Figure 1(b) there are 11 vertices, 5 faces, and 14 edges. Figure 2 shows some other drawings of knots and links in this style. Since such a graph is even-valent it will always have an Eulerian circuit, a tour which starts and ends at the same vertex and traverses each edge exactly once. However, the example in Figure 1 is special because for whatever edge one starts at, if one "cuts through" the 4-valent vertex one gets to at the end of this edge and continues in this way, this generates the graph shown. Not all 4-valent plane graphs whose faces are even sided have this property. For example, the graph on the left in Figure 2, which has the interesting property of being a polyomino, a collection of 1x1 squares where pairs meet edge to edge, while having an Eulerian circuit does not have a cut-through Euler circuit. This polyomino breaks up into three separate circuits and, hence, can be thought of as a projection of a link, using the cut-through point of view for traversing edges. One way to see that this drawing (on the left in Figure 2) can't be the projection of a knot is that it has a "column" of squares (and also a row of squares) whose end edges have 2-valent endpoints which lie in parallel lines.

Knot theorists have studied many facets of 4-valent plane graphs but typically have not looked at the "face vector" information associated with graphs of knots, rather concentrating on issues related to crossing numbers, Gauss codes and Rademeister moves. Here I will concentrate on questions related to the face vectors of projections of knots, where in addition to 4-valent vertices, 2-valent vertices have been added so as to make drawings with edges that meet at right angles possible. One theme here is the value of looking at the same geometric structure from different points of view. The knots and links in Figure 2 can be thought of as plane connected 4-valent graphs, polyominoes, non-convex equilateral polygons (concentrate on the edges which make up the infinite face of the plane graph) and self-intersecting plane polygons.

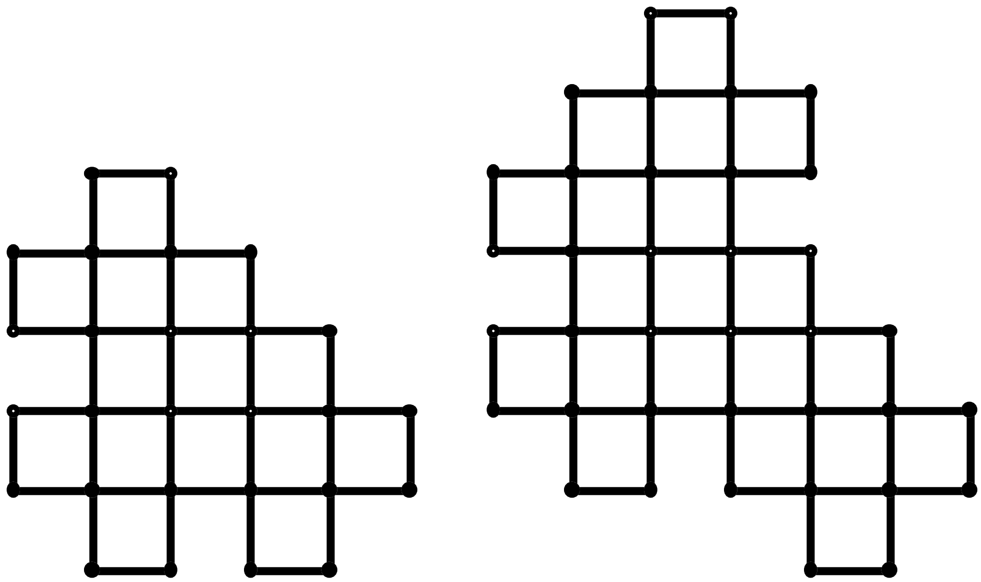


Figure 2 (The graph on the left represents 3 links and the one on the right a knot.)

One can draw diagrams which represent knots but which would not be polyominoes, as for example in Figure 3.

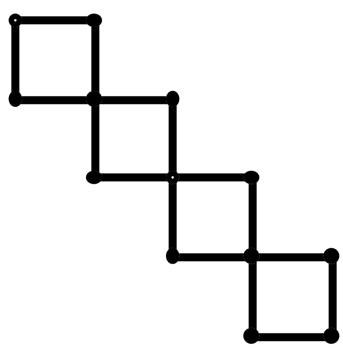


Figure 3 (Not a polyomino but cut-through Eulerian.)

Cut-through eulerian knot drawings

The drawings in Figures 1-3 are connected and even-valent, and thus they admit Eulerian circuits. An Eulerian circuit is a tour of the edges of a graph that starts and ends at the same vertex and traverses each edge of the graph once and only once. I am interested in a special type of Eulerian circuit known as a cut-through Eulerian circuit. The idea is that when one arrives at a 4-valent vertex along an edge, one always continues along the "middle" edge at the vertex, as suggested by Figure 4. As one arrives at the 4-valent vertex w from the edge at the right one "cuts through" w without turning to the right or the left at w. At a 2-valent vertex, when entering the vertex on one of the edges at it, one just continues on with the other edge. The diagram on the right in Figure 4 gives a more "global" view of a cut-through route in a typical 4-valent graph.

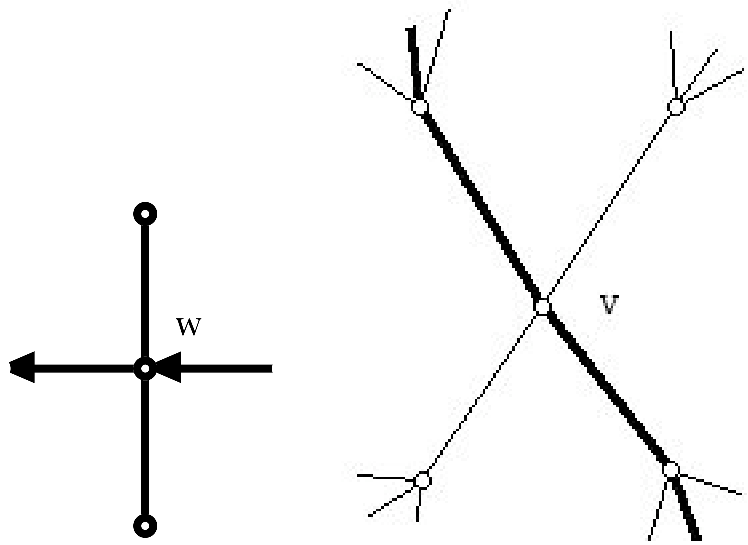


Figure 4 (Cut-through tour at a 4-valent vertex.)

Projections of knots have cut-through Eulerian circuits but they also have other Eulerian circuits as. The drawing on the left in Figure 2 is the union of three cut-through circuits but is not cut-through Eulerian. This drawing does not represent a knot but can be thought of as what knot theorists call a link with three components. The drawing on the right is cut-through Eulerian. Sometimes I will refer to a plane drawing of a graph with even-valent vertices and which is cut-through Eulerian as a CUTE embedding. I will also refer to CUTE polyomino drawings where all the bounded faces of the drawing define a plane graph which is a polyomino.

Figure 5 shows a CUTE drawing with 8 bounded faces. This drawing would correspond to a knot with 7 crossings. There are no knot drawings that are CUTE polyominoes which have fewer than 7 crossings. This drawing can be thought of as being part of a family of polyominoes which might contain CUTE drawings. One can think of the family as having a staircase of squares along two lines of slope 1 of "length 3" and a staircase of "length 2" along two lines of slope -1. The polyomino in Figure 5 will be denoted as the 2x3 staircase polyomino.

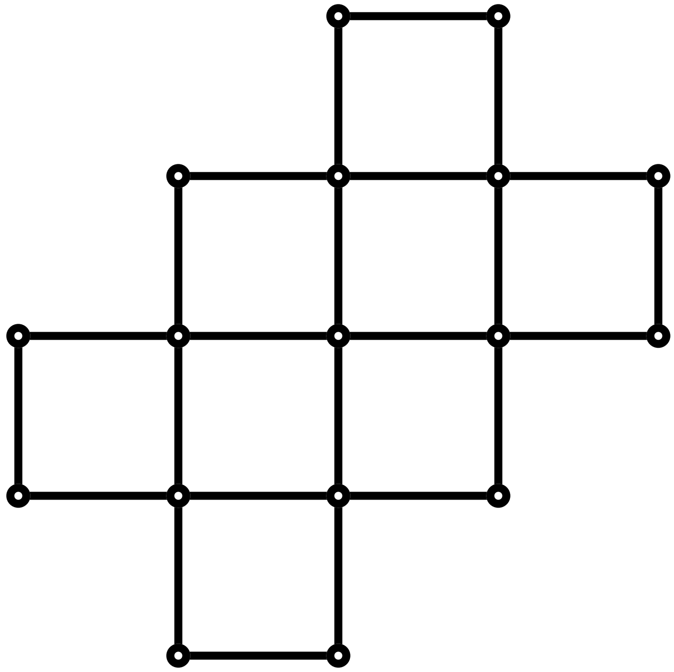


Figure 5 (2x3 staircase polyomino.)

It has 8 squares as does the "isomorphic" 2x3 staircase polyomino. Not all staircase polyominoes are CUTE. A nice "exercise" is to show that the mxn staircase polyomino has mn + (m-1)(n-1) 4-gons, and the size of the infinite face of such polyominoes when regarded as a plane graph is 2(2m-1) + 2(2n-1) (for m and n positive integers n and m). Figure 6 shows the 3x3 staircase polyomino which decomposes into three loops. In Figure 5 each row of squares is connected; I will refer to this as that the drawing is horizontally convex. It is also vertically convex. There are infinitely many CUTE drawings which are both vertically and horizontally convex.

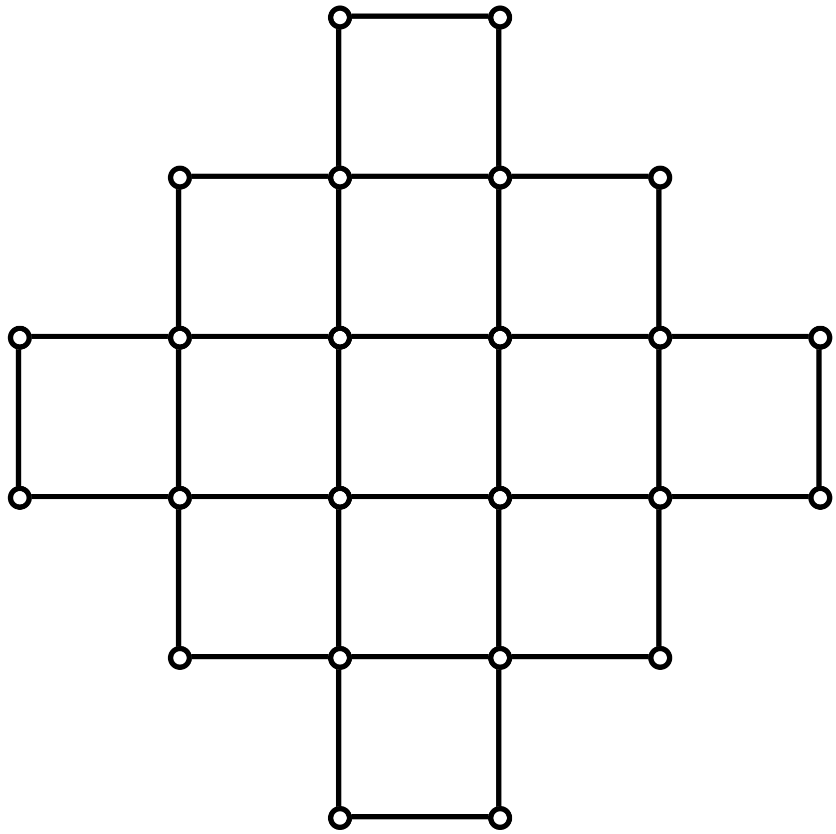


Figure 6 (3x3 staircase polyomino.)

It turns out that for an mxn staircase polyomino if the largest positive integer which divides both m and n is d (d, thus, is the greatest common divisor of m and n), then the polyomino breaks up into d simple circuits, the case d = 1 being when one has a CUTE drawing.

Not all 4-valent plane graphs which have cut-through Eulerian circuits can be represented as cut-through Eulerian polyominoes. Thus, the graph in Figure 7 has no such representation though as shown in Figure 7 it has a representation with rectilinear polygons with more than 4 bounded sides. The graph in Figure 7 is one of an infinite family of plane 4-valent graphs with exactly 8 triangles and some number of 4-gons (other than one). The graph of the regular octahedron is the member of this family with zero 4-gons but can't be drawn in a cut-through drawing. However, some graphs in this family can be drawn in the plane with a cut-through Eulerian circuit. The drawing in Figure 7 has this property and has three 4-gons.

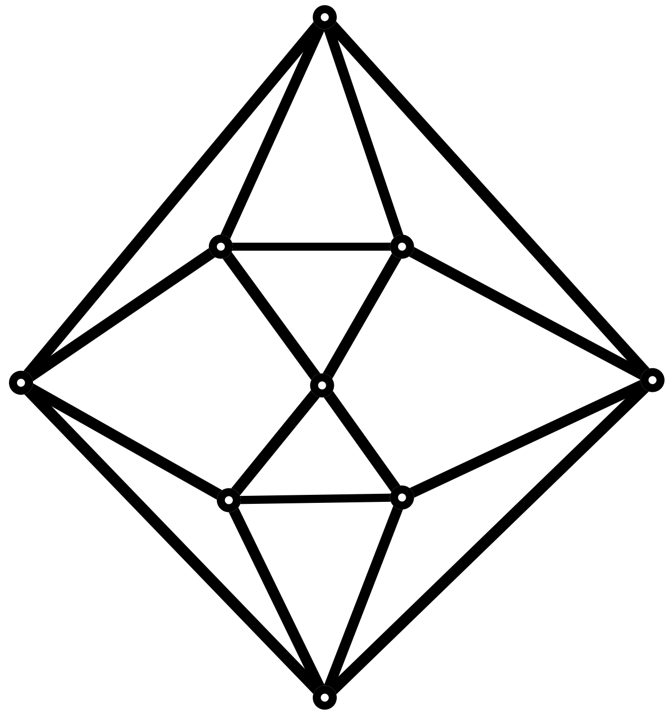


Figure 7 (Plane graph with 8 3-gons and 3 4-gons, including the infinite face which is a 4-gon.)

It is not difficult to see that there is no polyomino drawing (all bounded faces 4-gons) for this graph which has 9 crossings. Figure 8 shows a rectilinear polygon drawing of this graph where we have 9 4-valent vertices but 2-valent vertices have been added to achieve the rectilinear drawing. Remember that the same knot can have many different CUTE or rectilinear embeddings in the plane. Here I will not be pursuing this issue directly.

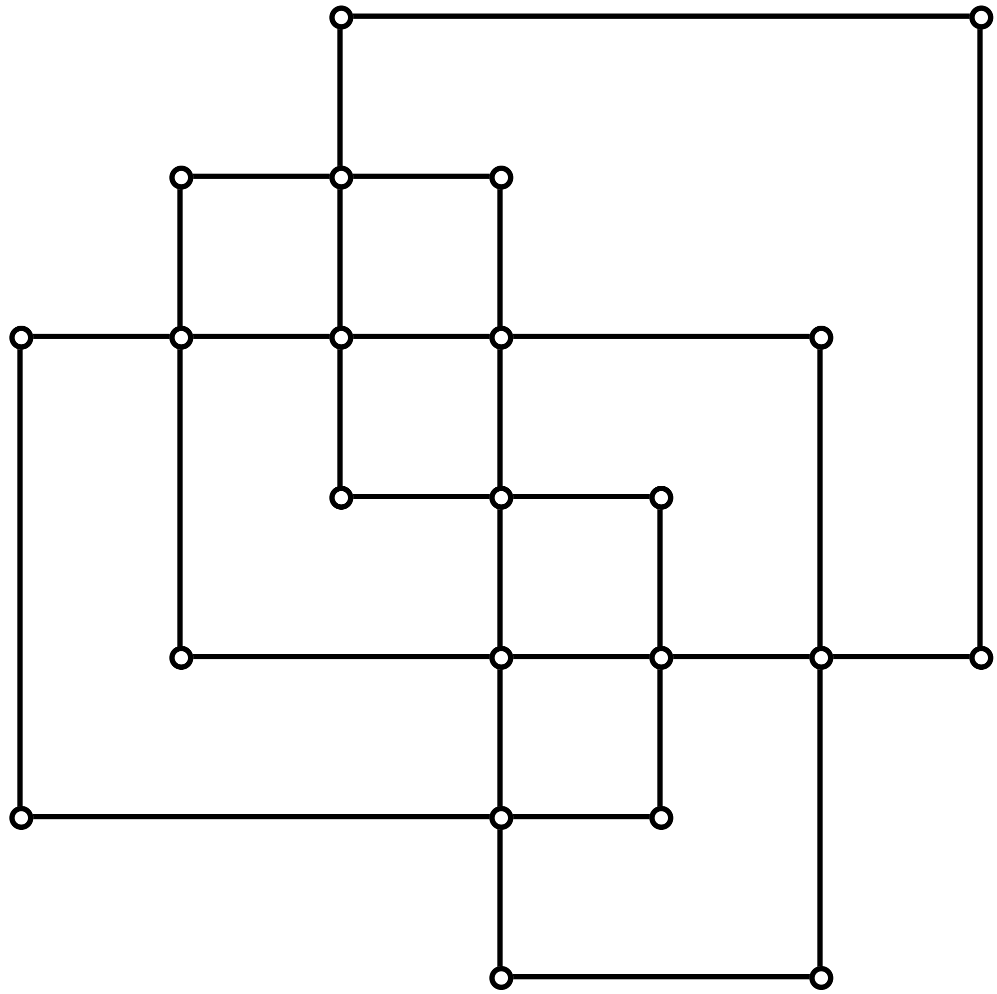


Figure 8 (Rectilinear polygon representation for the "knot graph" in Figure 7.)

The relatively small (not large number of 4-gons) examples we have looked at may have concealed that CUTE drawings can be very "irregular" and complex as we increase the number of 4-gons in the drawing. Figure 9, for example, shows a CUTE polyomino drawing that has thirty-one 4-gons. How can we represent or code such a drawing? One approach is to count the number of rows and columns of the drawing as a parameter of the drawing. Thus, in Figure 9 we have 9 rows and 8 columns. By comparison, Figure 5 has 4 rows and 4 columns while the polyomino on the right in Figure 2 has seven rows and 6 columns. Not surprisingly, this is a relatively crude, though interesting count. A more "quantitative" approach would be to record the number of cells in each row of a polyomino from top to bottom (row information) and the number of cells in each column of a polyomino from left to right. This will be done without noting the size of any gaps between the "runs" of cells, though in a more detailed approach the size of the gaps could also be recorded. Thus, the codes for the 3x2 staircase are:

Rows (horizontal code): 1, 3, 3, 1

Columns (vertical code): 1, 3, 3, 1.

For the more complex polyomino in Figure 9 we would write:

Rows (horizontal code): 1, (3,1), 5, (1, 3), 1, (1,3), 7, (3,1), 1

Columns (vertical code): 1, (3,1), (3,3), (1,3), (3,3), 5, (1,3), 1

Note that when there are several runs of cells in a row or column this is indicated by grouping these runs by size in a parenthesis.

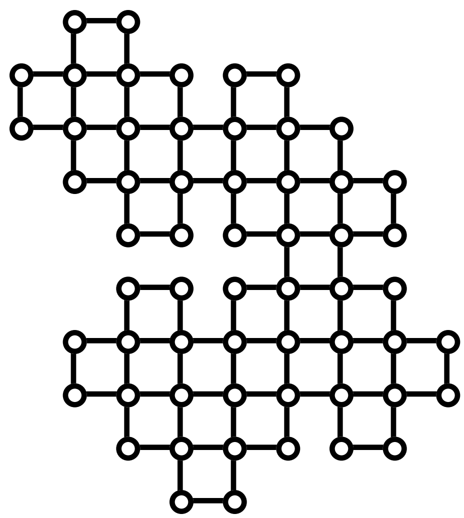


Figure 9 (Complex CUTE drawing.)

A necessary condition on the "blocks" in the code for a CUTE drawing is that each block be either an odd number or a "tuple" of odd numbers. However, I don't know of any way to determine for a specific code of numbers or pair of codes whether there is a CUTE drawing with this code.

Enclosing box

A CUTE drawing can be included in a rectangular box consisting of two horizontal lines that touch the upper edge of the drawing and two vertical lines that touch the left and right sides of the drawing. I will use r and s to denote the number of rows of the grid of an enclosing box and s to represent the number of columns of the grid of an enclosing box. In the spirit of earlier questions raised here, there is the issue of what values of p_4 are possible for an enclosing box of size rxs. It is particularly interesting to see the numbers of attainable p_4 for an enclosing rxr box. Note that the enclosing box can be thought of as a kind of "convex hull" for a CUTE drawing. Figure 10 shows an example of CUTE drawing with 7 rows and 6 columns and p_4 = 20.

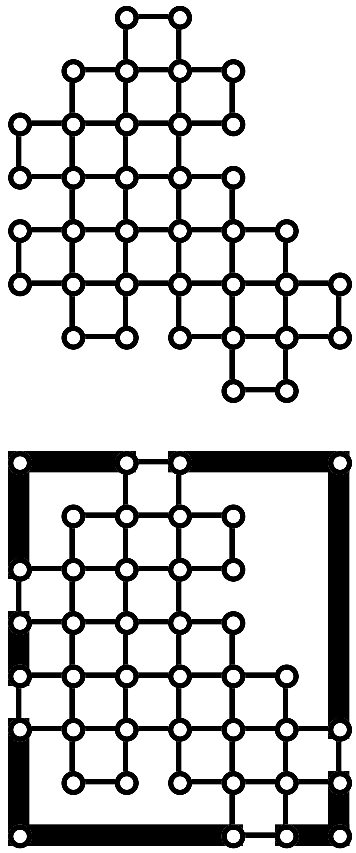


Figure 10 (A CUTE polyomino and its enclosing box; edges of the box, which are not part of the drawing are highlighted.)

The very dark lines in Figure 10 show the edges of the enclosing box that are not part of the original CUTE drawing. The number of squares within the enclosing box not in the CUTE diagram have 3, 1, 6, 1, 11 cells respectively. Note that as a "check" there are 22 cells inside the enclosing box but not part of the CUTE drawing, which with the 20 cells in the CUTE drawing adds to 42, which is 6x7.

Combinatorial considerations

In general many non-equivalent knots will have the same CUTE diagram. There are a variety of questions that can be posed about the nature of those knots which have a common CUTE diagram, arising from considering the pattern of "adding back" the under and over information about the projection of a knot that gave rise to the 4-valent plane graphs which we have been looking at here. However, in a variety of senses there are many different knots whose projections can be realized with CUTE drawings.

One purpose of this note has been to observe some properties of the numbers n, p_i and v_i defined above, as well as the number m, which represents the number of sides of a cut-through Eulerian rectilinear polygon when interpreted as a self-intersection polygon. Thus, while the infinite face in Figure 1 has 10 sides, viewed as a self-intersecting polygon, this drawing is an 8-gon. Sometimes perhaps it clarifies the ideas to write $F_!$ for the number of sides of the infinite face of the graph in Figure 1. In that example, $F_!$ is a 10-gon.

The diagrams we are drawing are connected plane graphs, so as noted earlier we can write V + F - E = 2 must hold for the number of vertices V, faces F and edges E for projections of knots or more generally links, where the bounded faces are 4-gons and the infinite face is an n-gon. However, we can also use some simple arguments to obtain other relationships. For example, since each edge has exactly two endpoints and bounds exactly two faces we have the following relationships when we have a polyomino which represents a knot or a set of links.

$2E = 2v_2 + 4v_4$

(counting edges using the valences of the graph)

which means

$E = v_2 + 2v_4$ (*)

and

$2E = 4p_4 + n$

(counting edges using the number of sides of the faces).

Recall that n counts the number of edges of the infinite face. The number of faces when the drawing is rectilinear will be the expression below:

$$F = 1 + \quad p_i$$

the "1" being to count the infinite face. The other faces will all be 4-gons in the case where we are dealing with a polyomino. In that case we can drop the summation and just write p_4. Thus, write $F = 1 + p_4$.

Putting the information from these various equations together we get the following facts:

$V + F - E = 2$

Since we have $V = v_2 + v_4$, $F = 1 + p_4$ and using (*) that $E = v_2 + 2v_4$, substituting in Euler's formula above we get:

$v_2 + v_4 + 1 + p_4 - v_2 - 2v_4 = 2$

Simplifying we have:

$p_4 = v_4 + 1$

In words this says that the number of crossings for a knot projection representable as a polyomino is one less than the number of 4-gons in the polyomino.

Also the number of vertices of our polyomino drawing viewed as a self-intersecting polygons obeys

$m = v_2$ and $v_2 = n/2 + 2$

so note, n must be even. In fact, n and m are related by: $n = 2m - 4$.

Again, for any drawing of the projection of a knot as a polyomino viewed as a plane graph the number of bounded face 4-gons is one more than the number of crossings of the knot in the particular projection involved. Furthermore, when viewed as a non-convex polygon, it is an equilateral polygon with $2m - 4$ sides. In Figure 3 we have $m = 10$ and the polygon has 16 equal length edges of length 1.

Theorem:

For k = 8 (smallest possible value for a CUTE drawing) and for any k at least 18 there is a polyomino (its vertices will be 2-valent and 4-valent when thought of as a graph) with k 4-gons that is cut-through Eulerian. In general, there can be more than one such polyomino for a given k.

Exploring polyomino drawings

For a polyomino P which is the union of s cut-through circuits (the polyomino is a projection of a link rather than a knot), one can add "plugs" of different sizes consisting of 4-gons to the drawing of P and which will combine cut-through circuits into bigger cut-through circuits. Thus a polyomino which is a projection of a link can be "extended" into a CUTE polyomino drawing by adding 4-gon cells which preserve the diagram being a polyomino. Figure 11 shows how to add an "L-shaped" triomino to the existing polyomino in Figure 6 to get a new polyomino with one fewer cut-through circuit. In this example, we had three cut-through circuits in Figure 6 and we have only two in Figure 11 after adding the cells with x's.

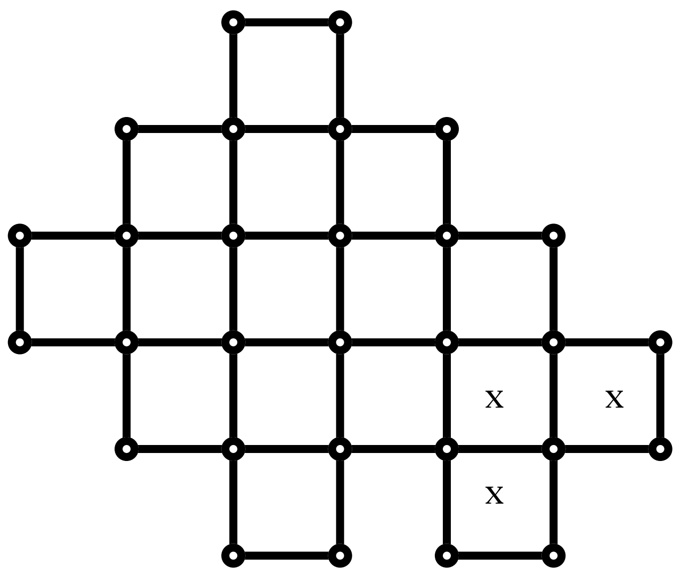


Figure 11 (The 3x3 staircase polyomino with three cut-through circuits can be transformed into a drawing with two cut-through circuts by adding additional squares.)

Now we can add two more 4-gons which "paste together" the two cut-through circuits in Figure 11 to obtain a CUTE drawing.

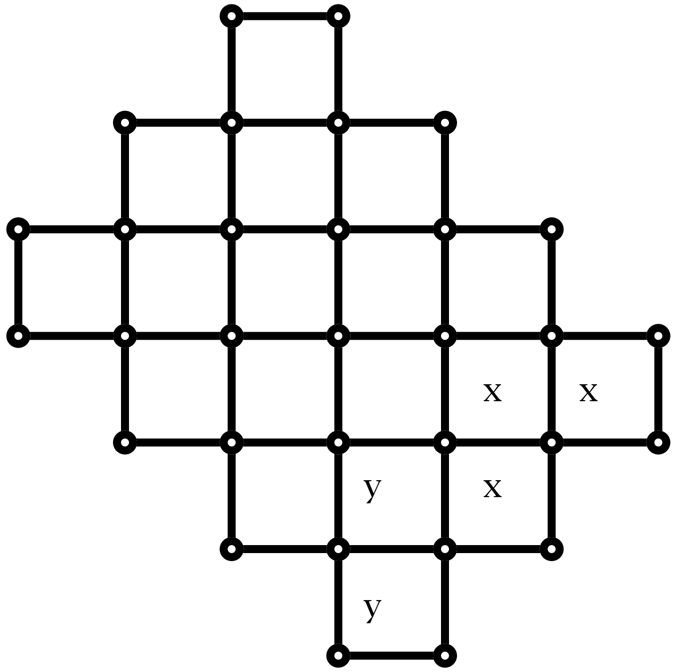
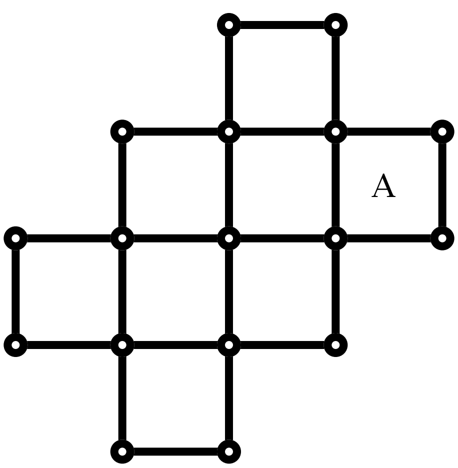
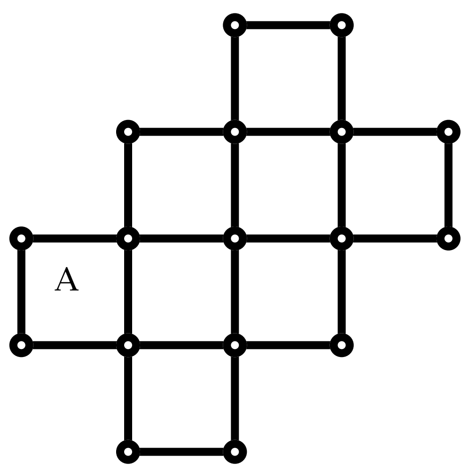


Figure 12 (By adding five 4-gons to the 3x4 staircase we obtain a CUTE drawing.)

After carrying out the two operations indicated in Figure 12 (the three cells labeled x and the two cells labeled y) we obtain the 3x4 staircase polyomino, which is CUTE. Other ways can be found to get new CUTE drawings from either unions of cut-through circuit polyominoes or pasting two CUTE drawings together.





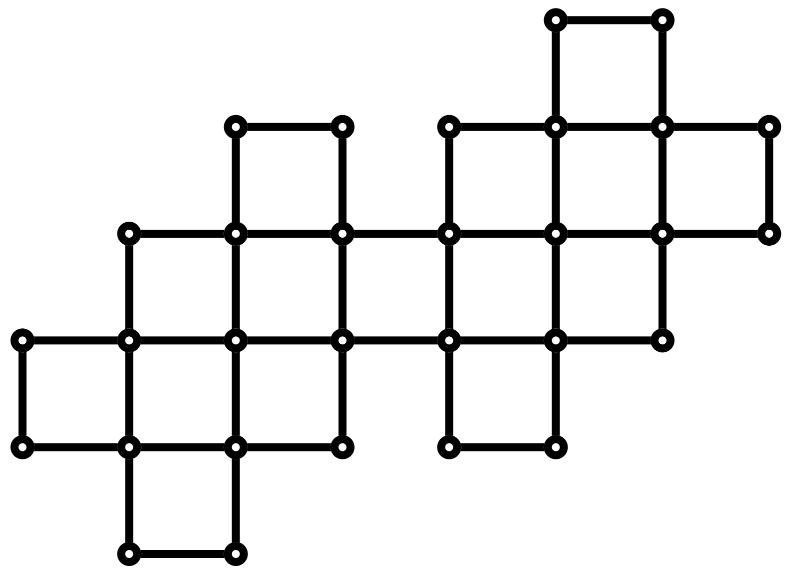


Figure 13 (By pasting CUTE drawings along a suitable edge we can obtain a "larger" CUTE drawing.)

In studying CUTE drawings it is natural to try to find operations done on CUTE drawings that will lead to another CUTE drawing with properties that are related to the initial drawing. Figure 13 shows an example of how to paste together two regions which are separately CUTE (by fusing them on the cell labeled A) to get a new CUTE drawing with more cells. The pieces each have eight 4-gons but the fused example has 15 4-gons. The construction in this case takes advantage of the fact that each 3x2 staircase (Figure 13, top) has edges with 2-valent vertices which are joined by an edge that can used to assist in this "pasting" process. I will call an edge which consists of two 2-valent vertices joined by an edge an exposed edge. Note that just as we can classify vertices of a CUTE diagram as to whether they are in the interior of the diagram or on the boundary, and by their valence, we can distinguish between three kinds of exposed edges.

a. Exposed edges which can appear anywhere in the CUTE diagram

b. Exposed edges which appear on the boundary of the bounding box for the CUTE diagram

c. Exposed edges which appear on one of the four edges of the bounding box of the CUTE diagram, but where there is exactly one edge joining two 2-valent vertices along this edge of the bounding box.

Mathematical explorations

In the mathematics that one is exposed to prior to college it may appear that there is no new mathematics to be discovered. While one can certainly argue that some "new mathematics" is more important than other "new mathematics" there is something wonderfully exciting about discovering new mathematics for oneself. The ideas looked at here, while rooted in work about knots from the past, are "new." Here are some explorations you might try out for yourself!

1. Which horizontal and vertical codes (see above) obeying the condition odd entries and tuples of odd entries give rise to CUTE polyomino drawings?

2. Let G be a plane 2-connected graph which is cut-through Eulerian. What is the smallest value of t such that there is a rectilinear plane drawing D of G such that the largest bounded face of D has at most t sides? Some such graphs G don't have CUTE drawings. Is there a "test" which allows one to distinguish which graphs G have CUTE polyomino drawings and for which ones must settle for rectilinear drawings?

An interesting special case of this question is for the plane 4-valent graphs with exactly 8 3-gons and all other faces 4-gons. This class is infinite and the number of inequivalent graphs with r 4-gons grows rapidly with r.

3. Investigate the number of exposed edges of different kinds that can occur in CUTE polyomino drawings with a particular number s of 4-gons.

4. How does the number of inequivalent CUTE polyomino drawings with s 4-gons grow as s graphs?

I have tried to argue that looking at knot projections from different points of view is an entry point into lots of interesting geometrical questions and offers lots of explorations for those who are well established as researchers in geometrical questions, but also for people who are beginners. While there is a vast literature on the theory of knots, I have not been able to find exactly this line of investigations in the literature. Enjoy your mathematical explorations!

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Addendum: (May, 2018)

This essay was submitted to the volume that was planned in conjunction with the MOVES III Conference sponsored by the Mathematics Museum. It was not accepted for publication but it may perhaps have some interest for people interested in recreational mathematics, knots, and research problems that can be worked on by high school and undergraduate students.

Using the table of "Unitary polyominoes" prepared by George Sicherman

https://userpages.monmouth.com/~colonel/nunitary/nunitary.html

it is possible to determine all the values between k equal to 8 to 18 for which there are CUTE drawings of polyominoes.

I welcome comments, and new questions and/or results related to these ideas.