The "Middle" of a Graph: Partitions, Eccentricities and Degrees of Vertices in a Graph

Joseph Malkevitch

Diagrams consisting of dots and lines (see Figure 1) called graphs are a very powerful visual tool for helping understand many topics within mathematics and applications of mathematics to other fields. In particular one can find the distance between the objects one represents using the dots in the graphs. This enables one to find vertices in the "middle" of graph in much the same way that a mean or median allows one to pick out a "middle" number for a collection of numbers. A middle vertex might be a good choice for locating a vaccination center.

Here I will look at a new family of research problems involving some well-known graph theory ideas, but building on a simple notion from number theory, namely the concept of a partition of a non-negative integer.

The number of partitions of n (positive integer) are the number of ways of writing n as a sum of positive numbers. For example, the partitions of 5 are {5}, {4, 1}, {3, 2}, {3, 1, 1}, {2, 2, 1}, {2, 1, 1, 1}, and {1, 1, 1, 1, 1}. Thus, there are 7 partitions of 5. Check for yourself that there are 11 partitions of 6.

While some definitions of a graph allow a pair of vertices to be joined by several edges or a single vertex to be joined to itself by an edge, that will not be allowed here. Figure 1 shows an example of a connected (one piece) plane (edges meet only at vertices) graph with 5 vertices and 5 edges. As we will see shortly, in this graph the vertices named 0, 1 and 4 have an equal claim to be called middle vertices (central vertices).

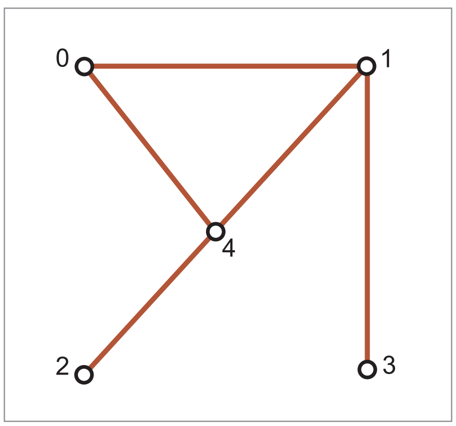

Figure 1 (A connected graph with 5 vertices and 5 edges.)

Vertices will be labeled with numbers and edges can be denoted using the vertices that are at their ends – thus, the edge joining vertices 4 and 1 in Figure 1 can be denoted 1,4 or 4,1. If u and v are vertices of a graph, a path from u to v is a sequence of edges where no vertex is repeated. Thus, in Figure 1, 2, 4, 1, 3 and 2, 4, 0, 1, 3 are paths from 1 to 3 but 2, 4, 1, 0, 4 is not a path from 2 to 4. The length of a path is the number of edges in it. A graph is connected if for every pair of vertices there is a path between them. If a connected graph has no "circuits" (a circuit is an edge tour with distinct edges that go from a vertex along edges back to the vertex the tour started at), then the graph is called a tree. Figure 1 is not a tree because 0, 1, 4, 0 is a circuit.

Suppose G is a connected graph. If two vertices of G are u and v, then the distance between u and v is the number of edges in a shortest path from u to v. The eccentricity of a vertex v in a connected graph is equal to the distance between v and some vertex w so that the distance between v and w is as large as possible. The eccentricity of vertex v measures the farthest that any vertex can be from v. I will use the notation $e(v)$ to denote the eccentricity of v. Also of interest for the vertices of a graph are their degrees or valences. The valence or degree of v, denoted $d(v)$, will be the number of local line segments that meet at v. (This definition allows one to use loops in the graph; however, for simplicity here I will consider only graphs without loops or multiple edges.) Also note there are special types of connected graphs of interest:

trees; 2-connected, planar; planar 3-connected, etc. (see Glossary; https://www.york.cuny.edu/~malk/Glossary.html).

For each vertex in Figure 1 we can write down its valence and eccentricity. Thus, $e(0) = 2$, $d(0) = 2$; $e(1) = 2$, $d(1) = 3$; $e(2) = 3$, $d(2) = 1$; $e(3) = 3$, $d(3) = 1$; $e(4) = 2$, $d(4) = 3$. So for this graph we have 3 vertices of eccentricity 2, 2 vertices of eccentricity 3 and 2 vertices of degree 1, 2 vertices of degree 3 and 1 vertex of degree 2. So this graph illustrates a graph with 5 vertices of partition type {3, 2} for eccentricities and partition type {2, 2, 1} for degrees. We can assign this graph a partition type based on pairs - the first entry of the pair is for eccentricity and the second for degrees. In this case the graph in Figure 1 is of type {3, 2}; {2, 2, 1}. The vertices of a connected graph that have minimal eccentricity are called central vertices. Central vertices in a graph can be thought of as being "in the middle" of the graph. However, note that there are graphs where all the vertices are central. We can raise general questions based on what we see here.

Figure 2 shows two different trees with 6 vertices. Can you verify for yourself that there are 6 different (non-isomorphic) trees with 6 vertices? The numbers at the vertices of the trees in Figure 2 show the eccentricities of these vertices. For a tree, there is either a single central vertex or there are two central vertices joined by an edge (theorem of Camille Jordan).

Warm up questions:

1. Suppose vertex w is the vertex in a connected graph farthest away from vertex v. Must vertex v and vertex w have the same eccentricity?

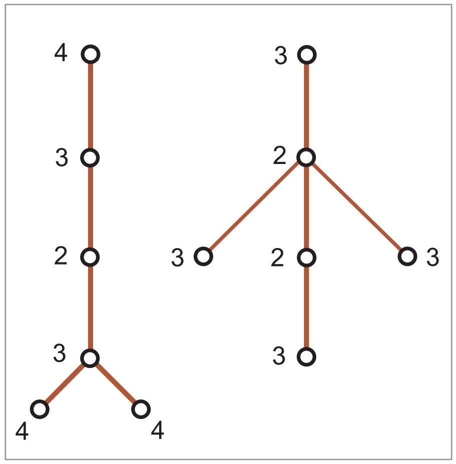

Figure 2: Two inequivalent trees and their eccentricities.

2. A connected graph G has 10 vertices. Can the partition of eccentricities of the vertices of G be the partition {4, 3, 2, 1} ? Can G be a tree? What happens for a graph with vertex eccentricity partition $\{n, n-1, ..., 3, 2, 1\}$?

Question 1

For connected graphs with 5 vertices

a. What degrees partition types are possible?

b. What eccentricity partition types are possible?

(Again for Figure 1's graph the degree partition type is {2, 2, 1} and the eccentricity partition type is {2, 2}.

Question 2

For connected graphs with 5 vertices if an eccentricity partition pair is possible, will the "reverse pair" exist? (For example we saw that (2, 2}; {2, 2, 1} exists. Does {2, 2, 1}; {2, 2} exist? Special cases: trees, planar graphs, and planar 3-connected graphs. How many different eccentricity partitions are attainable for a connected graph with n vertices and what is the largest number of parts possible? The partition {3, 2, 1} has 3 parts.

Question 3

For connected graphs with n vertices what degree partition types and eccentricity partition types are possible? Special cases: trees, planar graphs, and planar 3-connected graphs.

Question 4

For connected graphs with n vertices what eccentricity/degree pairs of partition types are possible? Special cases: trees, planar graphs, and planar 3-connected graphs.

Question 5

Explore the eccentricity/degree pairs for graphs where every vertex of the graph is central; that is, every vertex is in the "middle!" (If the graph has n vertices the eccentricity partition will be $\{n\}$!)

Note: The answers above for small values of n may be special and more general things may be noticeable as n grows.

Have fun working on these questions and let me know about any ideas you have and progress you make!

Reference:

West, D.B., 2001. Introduction to graph theory (2nd ed.). Upper Saddle River: Prentice hall.