Problems Related to 3-Polytopes (10/22/2021)

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Below are some relatively easy-to-state, I think accessible and interesting questions involving the graphs of 3-polytopes, which by Steinitz's Theorem are the planar 3-connected graphs. Notation: p(i) will represent the number of faces of a plane connected graph with i sides. Recall that for 3-polytopes all of whose faces are triangles the number of faces of the polytope must be even and that for 3-valent 3-polytopal graphs the number of vertices must be even. After each problem I list a few references which are related to that particular question.

1. Three infinite families of 3-polytopal graphs which are 3-valent (degree 3; cubic) that have attracted some attention are:

a. p(3) = 4; p(6) = h (h even) (h=0, tetrahedron)

b. p(4) = 6, p(6) = h (h not 1) (h=0, octahedron)

c. p(5) = 12 p(6) = h (h not 1) (Note: sometimes these graphs are called fullerenes.) (h=0, icosahedron)

These families of graphs define three infinite families of graphs which have all triangular faces by constructing the dual graphs of the ones above. Such graphs have h 6-valent vertices. Note that for a particular graph in such a family there are many metric realizations of the graph. Let P* denote a fixed metric realization of one of these graphs.

Do all the triangulations in these three families have a spanning tree such that when edges of the spanning tree of P* are cut, the faces of P* can be "opened up" so that the faces can lie in plane without overlap. (Sometimes such an unfolding into the plane is called a *net*.)

Comments:

a. The questions above represent special cases of a problem I associate with the work of Geoffrey Shephard (1927-2016), whose general case is still unsolved:

Does every convex 3-polytope have a spanning tree whose edges when cut allows one to unfold the polyhedron so its collection of faces as polygons in the plane do not overlap?

Cutting edges of a polyhedron to unfold its faces flat in the plane is a notion that goes back to Albrecht Dürer (1471-1528).

b. Even for some metrical tetrahedra one can choose a spanning tree where the unfolding does overlap but for these tetrahedra the choice of a different spanning tree will allow a non-overlapping unfolding. For specific 3-polytopes with many vertices, finding a non-overlapping unfolding by cutting edges of a spanning tree is a bit like finding a needle in a haystack but in all cases so far where a non-overlapping unfolding has been sought, it has been found.

References:

Grünbaum, B., Klee, V., Perles, M.A. and Shephard, G.C., 1967. Convex polytopes, New York: Interscience. (There is a revised and updated version of this book from 2003, published by Springer. Chapter 13 is the Chapter relevant to the problems here in both versions.)

Demaine, Erik D., and Joseph O'Rourke. Geometric folding algorithms: linkages, origami, polyhedra. Cambridge University Press, 2007. (The paper edition updates the hardcover edition.)

O’Rourke, J., 2011. How to fold it: the mathematics of linkages, origami, and polyhedra. Cambridge University Press.

2. Given a plane triangulation with at least 4 vertices T, let isos(T) = m (m at least 2; m can be set to ∞) means that T can be realized metrically with m but not fewer distinct edge lengths, where all of the triangles in the realization are strictly isosceles. (m is set to infinity when there is no isosceles triangle realization for an polyhedron which has the combinatorial type of T.)

(Historical setting: Finding natural families of "symmetrical" polyhedra has engaged geometers since Euclid. Such families include the Platonic solids, the Archimedean solids and the Johnson solids (regular faced polyhedra). Norman Johnson (1930-2017) pioneered the study of 3-dimensional convex polyhedra all of whose faces are regular polygons, as a way to move beyond the study of the Platonic and Archimedean solids. One natural way to extend this approach is to look at convex 3-dimensional polyhedra all of whose faces are strictly isosceles triangles, or whose faces have a mixture of isosceles and equilateral triangles. The problem below is inspired by this "progression" of work.)

a. Study the behavior of icos(T) for specific plane triangulations T in an effort to understand the behavior of this invariant of a plane triangulation. The case m = 2 is of particular interest. There are many plane triangulations which can't be realized with congruent strictly isosceles triangle polyhedra.

b. Study the behavior of isos(T) for the families of triangulations obtained as duals of three classes of graphs in Problem 1 above.

David Eppstein has shown that there are (infinitely many) triangulations where there is no metrical realization with finitely many lengths so that all its triangles are strictly isosceles. (For this case, set isos(T) = ∞.)

Given a plane triangulation with at least 4 vertices T, let isosEqui(T) = m (m can be set to ∞) means that T can be realized metrically with m but not fewer distinct edge lengths, where all of the triangles in the realization are equilateral or strictly isosceles triangles.

Note: When m = 1 it is known that there are exactly 10 combinatorial types of convex 3-dimensional polyhedra whose faces are all equilateral triangles. These are often referred to as the convex 3-dimensional deltahedra. The best known of these are the regular tetrahedron, regular octahedron and regular icosahedron.

References

Eppstein, David. "On Polyhedral Realization with Isosceles Triangles." Graphs and Combinatorics (2021): 1-23.

Grünbaum, Branko. "A convex polyhedron which is not equifacettable." Geombinatorics 10.4 (2001): 165-171.

Malkevitch, Joseph. "Milestones in the history of polyhedra." Shaping space. Springer, New York, NY, 2013. 53-63.

Malkevitch, Joseph. "Convex isosceles triangle polyhedra." Geombinatorics 10 (2001): 122-132.

3. While there are only 10 combinatorial types of convex 3-dimensional polyhedra whose faces are all equilateral triangles, for each of the 10 types there are many different realizations, each with a different edge length. However, for a fixed combinatorial type all the metrical realizations are related by a similarity transformation. Intuitively, different such polyhedra can only differ in "size."

Problem: Explore what convex, as well as non-convex 3-dimensional polyhedra exist whose faces are unions of regular polygons.

Comments:

a. It is particularly interesting to study convex 3-dimensional polyhedra which are unions of:

i. equilateral triangles

ii. squares

iii. squares and equilateral triangles

iv. regular pentagons, squares and equilateral triangle.

It is of interest to study the unfoldings of such polyhedra. There are convex 3-dimensional polyhedra whose faces are unions of equilateral triangles, for example, an anti-prism with two regular hexagons and 12 equilateral triangles which are not convex deltahedra. Unfoldings of these (allowing cuts along the edges that decompose the faces into regular polygons) are a subset of the polyiamonds, clusters of plane equilateral triangles that meet edge to edge.