Geometric Problems Involving Partitions

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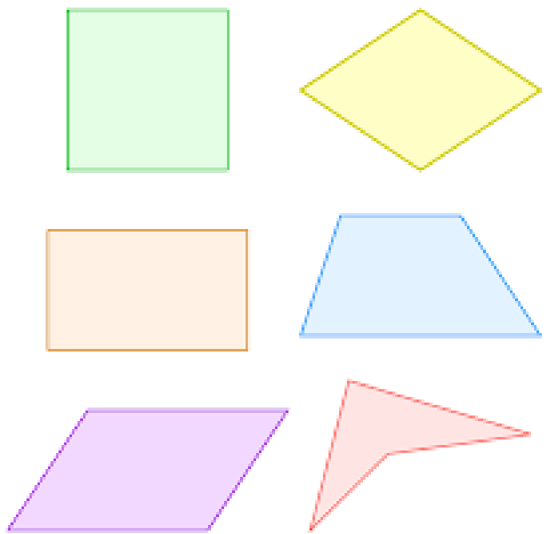
Dedicated to the memory
of Branko Grünbaum
(1929-2018)

One of the great discrete geometers
of the late 20th - early 21st century

(Died, Sept. 14, 2018)

What are the names for these shapes?



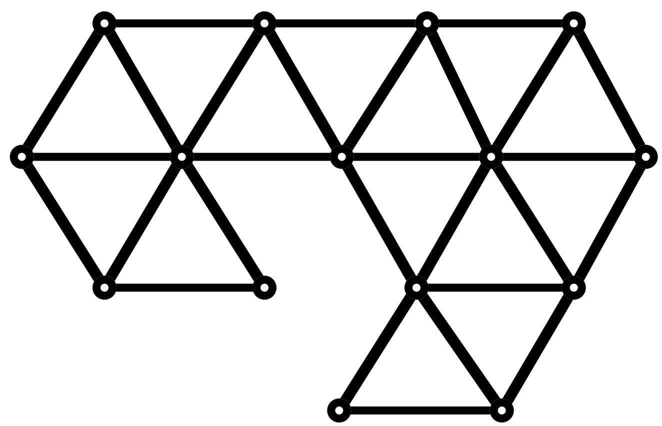
Are isosceles triangle, rhombus, rectangle, parallelogram *sensible* names for convex plane polygons?

Classifying quadrilaterals

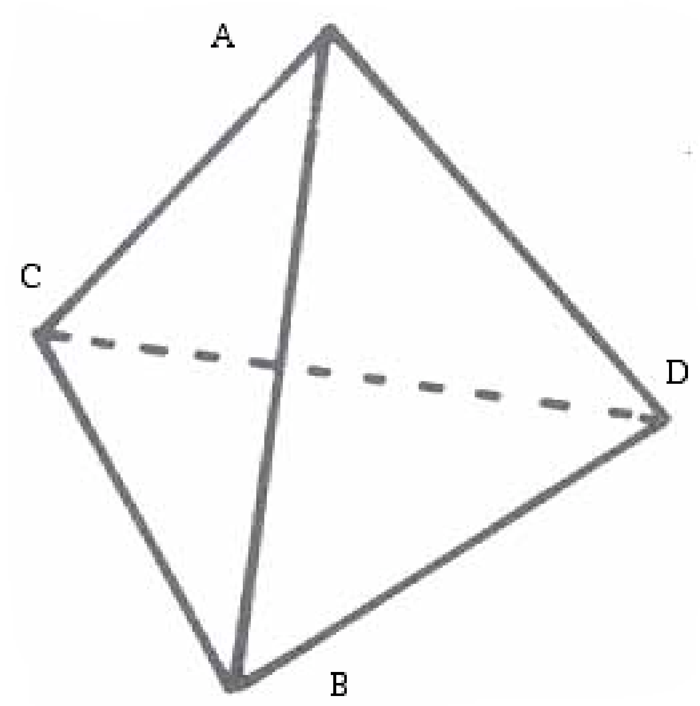
How many different kinds of convex quadrilaterals are there?

Convex means no holes or dents:

Non-convex polyiamond:



Classifying tetrahedra



How can one give names to different kinds of tetrahedra?

Partitions

Given a positive integer n, its partitions are those positive integers that sum to n.

Partitions of 3:

{3}

{2, 1}

{1, 1, 1}

For a triangle, 3 sides, we will interpret the partition {2,1} to mean that the triangle has two sides of one length, and one side of another length. Usual name: an isosceles triangle.

From this perspective (side length partitions) there are 3 kinds of triangles taking into account side lengths.

All three kinds "exist."

{3}

All sides equal, an equilateral triangle

{2, 1} - isosceles triangle

{1,1,1} - scalene triangle

But convex polygons have angles as well as side lengths.

For a triangle the angle partition {2, 1} will mean a triangle with two angles of one measure and 1 angle of another length.

So potentially, there are 3x3 or 9 possible types of triangles, but only these exist:

{3} ; {3}, {2,1} ; {2,1}. {1,1,1} ; {1,1,1}

These are our "friends," the equilateral, isosceles and scalene triangles.

More generally, for a convex n-gon we will assign an ordered pair of partitions, one for side lengths, and one for angles:

Example: {3, 1} ; {4}

would mean a convex quadrilateral with 3 sides of one length, and one side of another length and all four angles of the same measure.

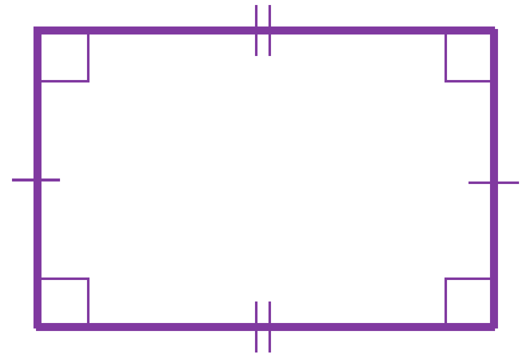
Can you see that in the Euclidean plane there can be no such quadrilateral!

Proof:

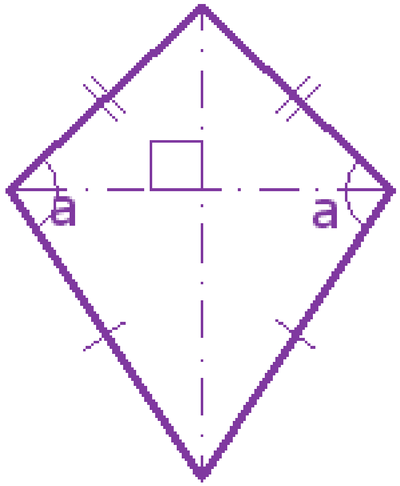
{4} would mean all 4 angles would be right angles, and the only possible side lengths partitions are {2, 2} (rectangle) and {4} (square)

Note: Nothing is said here about relative sizes of the edge lengths or "adjacent" vs. "opposite" sides of a polygon (even-sided case).

Rectangle:



Kite:



This angle/side length pair approach is originally due to Branko Grünbaum but it is my modification to look at this from a partitions viewpoint.

Partitions of 4:

{4}

{3, 1}

{2, 2}

{2, 1, 1}

{1, 1, 1, 1}

One can extend the partition notation to capture what Grünbaum did by addition of O (opposite) or A (adjacent) as a "decoration" on the partition notation.

{2, 2}O; {4}

{2,2}A; {2, 2}O

Of the 49 possible types Grünbaum showed that 20 existed but he failed to note that

$$\{2, 1, 1\}O; \{2, 1, 1\}O$$

exists, (so 21 types) as shown by Orlando Alonso (Lehman College) in his EdD Thesis at Teachers College.

Alonso also showed that one could also "decorate" the angles partitions according as they were acute, right or obtuse, and obtained the fact that there are over 100 different "types" of convex quadrilaterals using this point of view.

Theorem (Grünbaum-Alonso):

If in the table (side/angle) of 49 types or 25 partition types there is an entry in the i, j position there is an entry in the j, i position!

Note: The "error" above was on the diagonal so this did not affect the theorem.

Comments:

Grünbaum calls this *side-angle reciprocity*.

Alonso extended the analysis to concave quadrilaterals and self-intersecting quadrilaterals and the "reciprocity" result does not hold there. Convexity seems crucial.

Branko Grünbaum:

The angle-side reciprocity of quadrangles, Geombinatorics, 4 (1995) 11-16

Orlando Alonso

Enumeration of self-intersecting and simple concave quadrilaterals via paired partitions.

Geombinatorics, Jan. 2013.

Grünbaum conjectured that side-angle reciprocity held for his general approach for n-gons but a counterexample was found.

However:

Conjecture: (Malkevitch, Alonso)

The partition type version of side-angle reciprocity holds for n-gons.

Note: The counterexample to the general conjecture disappears for this more restricted conjecture.

Comment:

Partitions can't be counted in closed form but there is a *not* widely know "generalization" of partitions called "compositions" (ordered partitions) that can be counted easily!

The partition {2, 1, 1} gives rise to (2, 1, 1), (1, 2, 1) and (1, 1, 2).

Research problem for undergraduates:

Enumerate the convex pentagons by side/angle partitions and determine for each partition type if their is a convex polygon which tiles the Euclidean plane.

(Note: All edge-to-edge pentagonal convex tilers are now known. Very recent computer enumeration.)

Derege Mussa (from Ethiopia, and a Teachers College Ph.D.) showed that all of the partition types by side length (11 kinds) exist, and an an extension taking edge patterns into account leads to 25 types.

Open research problem:

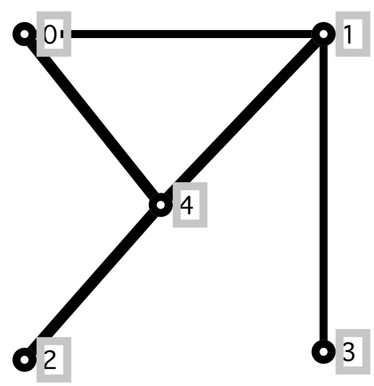
For each of the 11 types (or 25 types) of tetrahedra is there a representative of this type that tiles 3-dimensional space?

Degree and eccentricity partitions for a graph:

A graph is a diagram consisting of dots (vertices) and curves (line segments) joining up the dots (edges).

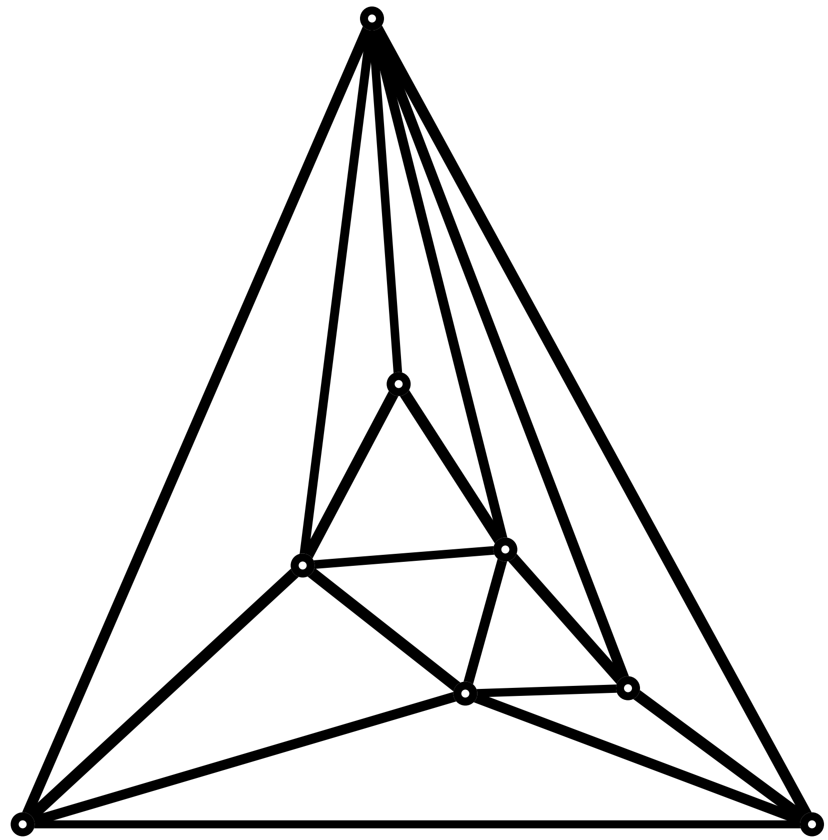
(no edges joining a vertex to itself, or more than one edge joining two vertices)

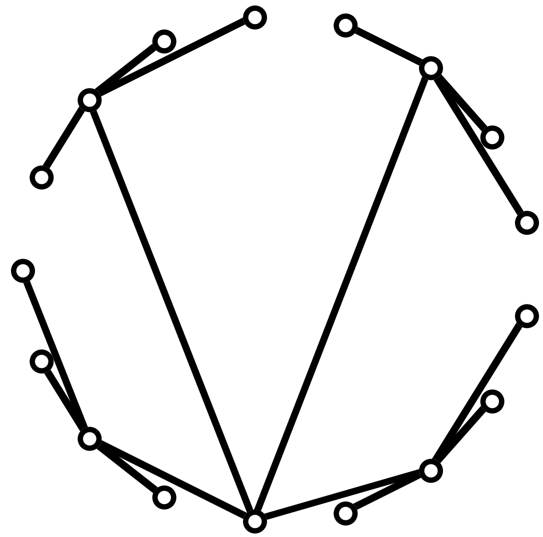
The graph below has 5 vertices and 5 edges, the names of the vertices are 0, 1, 2, 3, and 4.

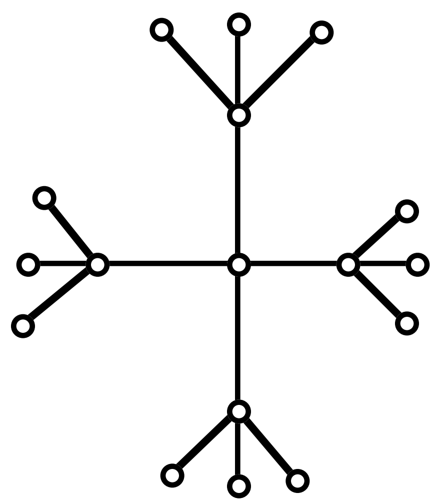


The degree (valence) of a vertex is the number of local curves at the vertex.

The distance between two vertices is the length of the shortest "walk" along edges between the vertices.

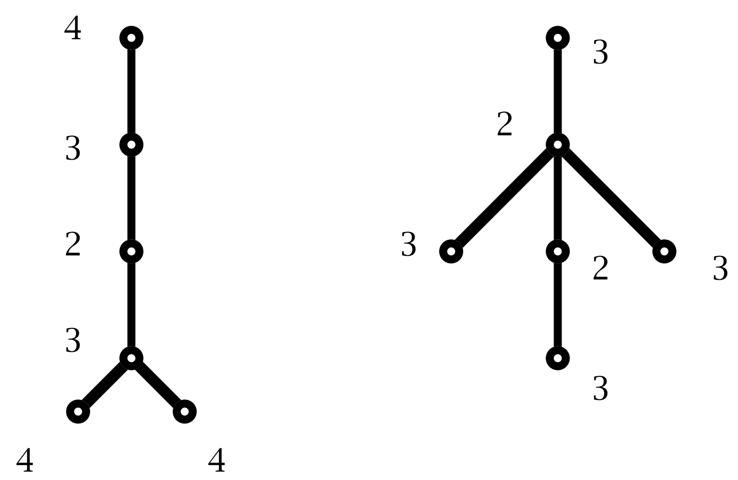






The eccentricity of a vertex v is the farthest that any vertex can be from v.

For these two graphs (trees) each with 6 vertices the eccentricity of the vertex is used as a label.



Research question:

Enumerate the degree partitions, eccentricity partitions of graphs with n vertices. Enumerate the degree partition, eccentricity partition pairs of graphs with n vertices.

Special case: trees; triangulated polygons; polyiamonds

Thanks for your attention!

Comments?

Questions?