The Transition from Two Dimensions to Three Dimensions - Some Geometry of the Tetrahedron

Prepared by:

Joseph Malkevitch
Department of Mathematics and Computer Studies
York College (CUNY)
Jamaica, New York 11451

Derege Haileselassie Mussa
Department of Mathematics and Statistics
Hunter College (CUNY)
New York, New York 10065

Given three (straight) sticks, when can they be used to make a triangle? Given six (straight) sticks, when can they be used to make a tetrahedron?

The triangle is the simplest example of an important kind of 2-dimensional geometrical structure - the polygon. The tetrahedron is the simplest of an important kind of 3-dimensional geometrical structure - the polyhedron. In this note we will look at how some ideas involving basic combinatorics, graph theory, and geometry can be used to help get insight into the thinking about the geometry of the tetrahedron and the transition between thinking about 2-dimensional problems to thinking about 3-dimensional problems. Along the way we will see lots of opportunities to provide students with problems for thinking about geometrical questions which have not been well explored. The "methods" here develop further the ideas for classifying quadrilaterals previously discussed in Alonso and Malkevitch [2,3] which in turn built on work of Branko Grünbaum [4}. We will try to keep the development of the ideas as informal as possible. By a triangle we will mean 3 points in the plane not all on a line. By a tetrahedron we will mean 4 points in 3-space not all situated in one plane. We shall be interested in two types of questions. When does a triangle or tetrahedron with specific edge lengths exist? How can we classify triangles and tetrahedra in a natural way?

In this note the diagrams we draw will not be drawn to scale and are designed to help one in an informal way think about the objects involved.
Figure 1 shows a general triangle and one with specific "proposed" lengths.

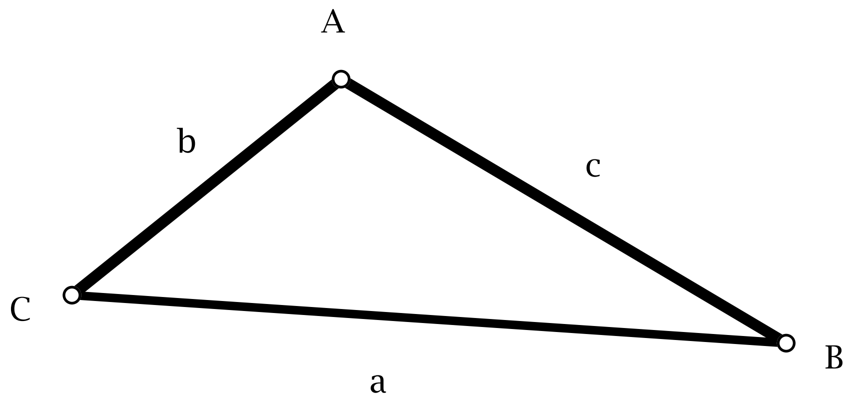

Figure 1a (A labeled triangle with general side lengths)

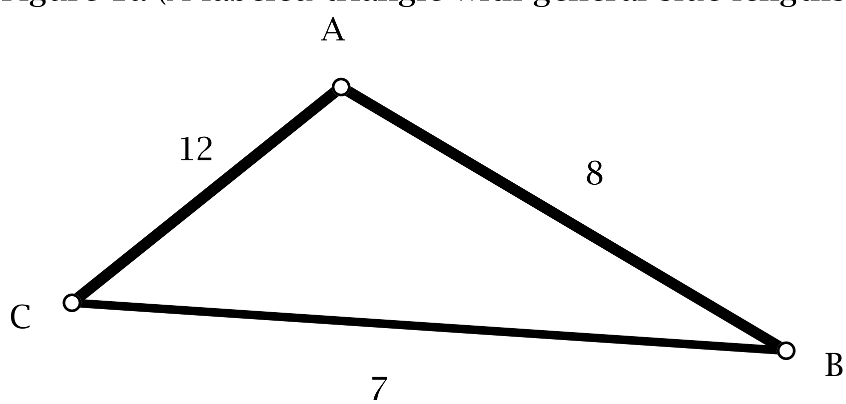

Figure 1b (A triangle with side lengths 7, 8, and 12)

Does such a triangle really exist physically? The answer is given by the triangle inequality. If the sum of any two of the side lengths for the triangle exceeds the length of the third side, the triangle exists. Thus, there is a physical triangle with the lengths shown in Figure 1b. If one interchanges two of the lengths there is still a physical triangle with these lengths; all of the triangles with any of the 6 ways (the first side length can be chosen in 3 ways, the next in 2, and the last in one way) to label the edges in Figure 1b exist and are congruent to each other! We will see that nothing this "nice" happens for tetrahedra. Note that if one has three numbers where the sum of two equals the length of the third, 7, 8, and 15, one can find three points on a line where these lengths are the distance between the pairs of the points. We have 3 collinear points, which can be thought of as a "degenerate" triangle. The definition of triangle is not "continuous" with respect to moving the point A with respect to side BC, a very small change can take a triangle and transform it to a degenerate one.

What different kinds of triangles are there? One approach to this is to use the notion of a partition of an integer n. The partitions of n are the ways that one can write n as the sum of non-negative integers. The partitions of 3 are {3}, {2, 1}, and {1, 1, 1}. Note that since we are using set notation, {2, 1} and {1, 2} are considered the same partition of 3. We will interpret {2, 1} as meaning we have a triangle where two edges have equal length, and the third side of the triangle has a different length from the other two. These partition classes correspond to the more familiar designations of equilateral triangle (E), isosceles triangle (I) and scalene triangle (S). We will use these terms when we classify tetrahedra, whose faces are, after all, four triangles. For details about how to use partitions to classify convex quadrilaterals and related problems, see Alonso [1] and Alonso and Malkevitch [2, 3].

Figure 2a and Figure 2 show two ways to represent a tetrahedron on a flat piece of paper. While the hidden line drawing in Figure 2a is more suggestive of something 3-dimensional, we will rely on the style of diagram in Figure 2b because of its connection with graph theory.

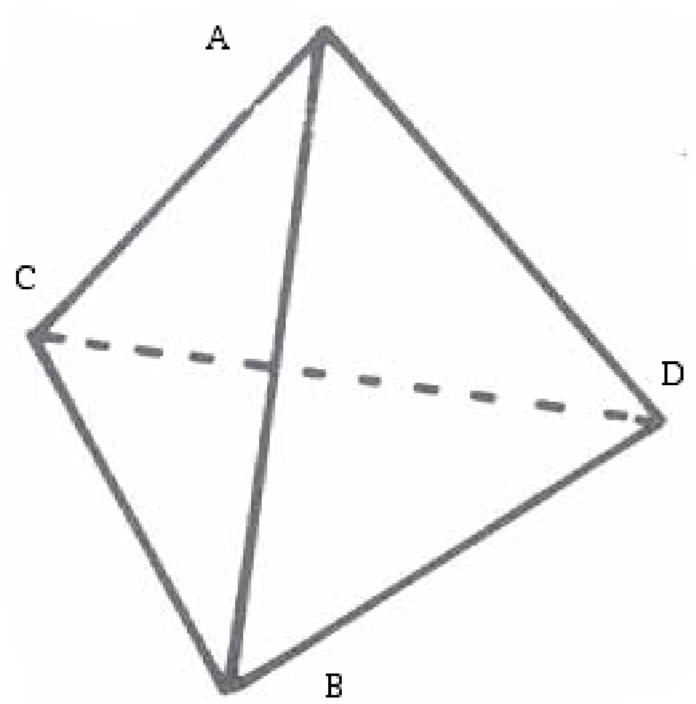

Figure 2a

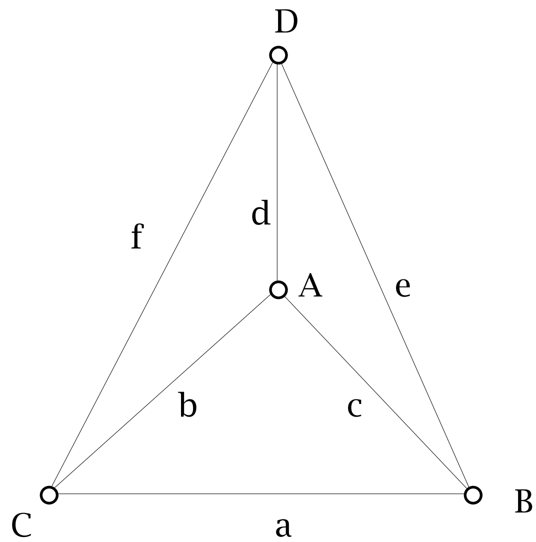

Figure 2b

Figure 2 (Two ways to represent a 3-dimensional tetrahedron in the plane)

Given an ordered sextuple of 6 numbers (repeats allowed) we will use the notation S(a,b,c,d,e,f) to denote the way of labeling the edges as shown in Figure 2b. To illustrate how this notation works, the potential tetrahedron in Figure 3a is denoted S(11,5,12,7,8,9), while S(11,8,12,7,5,9) would give rise to the drawing in Figure 3b.

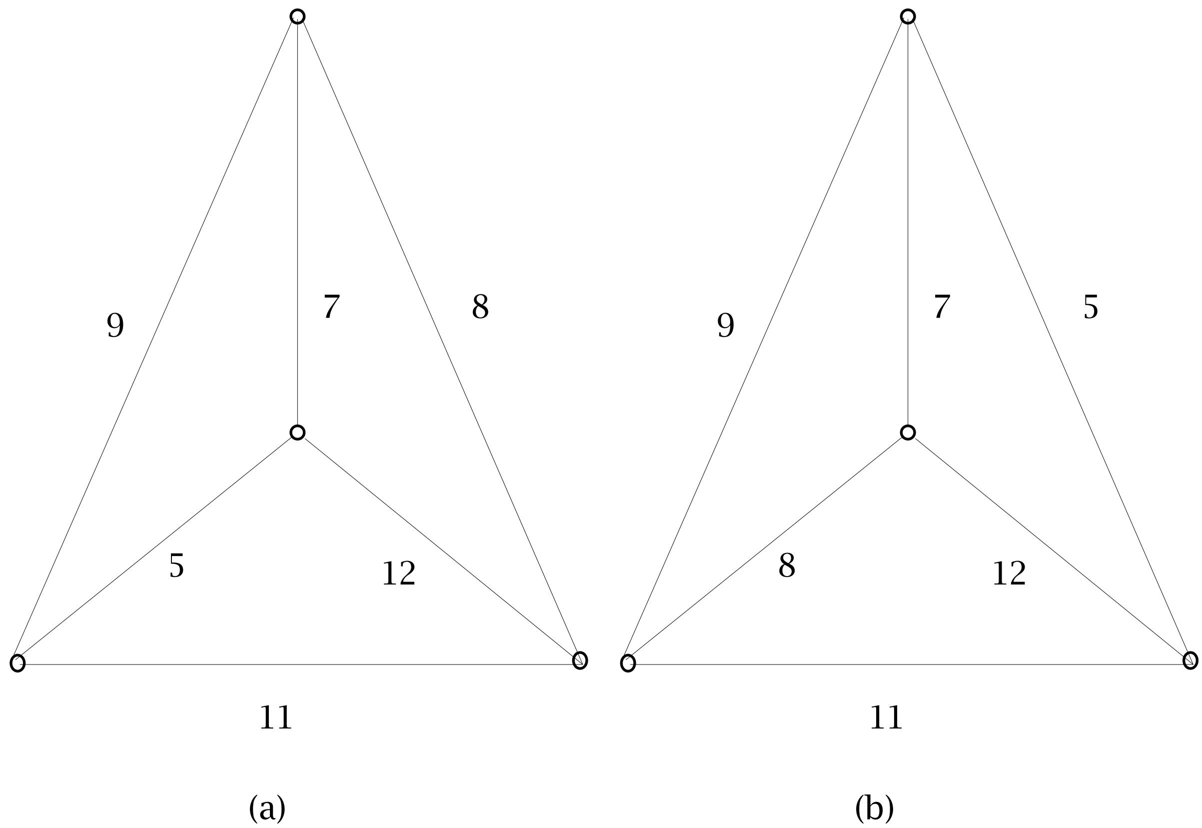

Figure 3 (Two potential tetrahedra with the same edge lengths but differently configured)

The two labelings are related by interchanging the lengths on a pair of edges. For a specific choice of 6 values, will an actual tetrahedron exist? What is the largest number of incongruent tetrahedra with 6 different edge lengths that there can be if there is at least one physical tetrahedron with these edge lengths? Before addressing these questions, note that if all of the 6 values a, b, c, d, e, and f are different, there are 720 different ways to label the edges in Figure 2b. Could all of these 720 different labelings lead to 720 non-congruent tetrahedra? Viewed as a graph - a diagram consisting of vertices (dots) and edges (line segments) drawn without edges crossing in the plane, the diagram in Figure 2b is the complete graph (every vertex joined to every other) on 4 vertices. This graph has 4 vertices, 4 faces (regions), and 6 edges. For this graph there are 24 ways (4!) to map the vertices to themselves which preserve the edges. These 24 automorphisms of the graph (mappings of the vertices to vertices that preserve edges) mean that for each labeling of Figure 2b there are 24 other labels which give rise to exactly the same set of faces.

Problem 1:

Find the 24 other ways to label the edges of the diagram in Figure 1b with the letters a, b, c, d, e, and f so that one gets the same set of faces for the tetrahedron that this diagram has the potential to represent.

Dividing 720 by 24 means that there can be at most 30 different (incongruent) tetrahedra with the same collection of 6 different edge lengths. (Is this the number you thought when the question was first raised above?)

Problem 2:

Verify that it is possible to choose 6 different integer lengths and construct 30 incongruent tetrahedra with these lengths. (For the very patient you might construct a complete set of these 30 tetrahedra using a scale for the edge lengths that produces nice physical models!) Note that for each of the different 30 possible tetrahedra there are 24 different ways to label the edges in Figure 2b to obtain the same set of faces with the given edge lengths.

Problem 3:

While the lengths as shown in Figure 3a allow one to construct a physical tetrahedron, can you see a simple argument that show there is no physical tetrahedron with the lengths shown in Figure 3b?

Now, what about how to tell when a tetrahedron exists, given a proposed collection of lengths for its six edges. In light of the fact that one can't have a triangle unless the strict triangle inequality holds, a natural necessary condition for a tetrahedron to exist is that for the edges that form the triangular faces one would want the triangle inequality to hold. Perhaps surprisingly, it is not hard to find examples where the triangle inequality holds for the edges forming each of the four faces of the proposed tetrahedron but the tetrahedron does not exist.

For example, S(4, 4, 4, 7, 4, 4), even though the triangle inequality holds for every triple of three numbers, when used to label the diagram in Figure 2b, does not exist. To see this provides a non-standard opportunity to look at some geometry and trigonometry. From Figure 4 we see that we have two equilateral triangles that would share an edge. If we think of two such triangles as lying flat in a plane, we can ask if we can flex the triangles about the edge CB so that we get a tetrahedron with the required edge lengths 4, 4, 4, 4, 4, and 7.

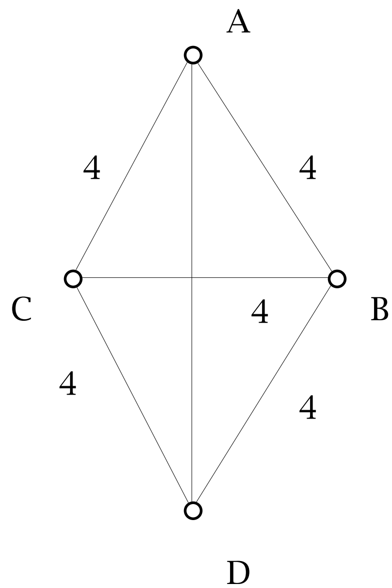

Figure 4 (Diagram to help show that there is no tetrahedron with five edges of length 4 and one of length 7)

If AB, AC, BC, DC, and BD are all of length 4, what is the length of AD? We can figure this out in a variety of ways, but let us use the law of cosines! We can compute the length of AD in terms of AC, DC and the angle at C by using triangle ADC. What is the angle at C? Since two equilateral triangles share the edge CB, the angle at C is 120 degrees. Thus, we can write, using the law of cosines, that:

$AD^2 = AC^2 + CD^2 - 2(AC)(CD)\cos(120^\circ)$. Now AC is 4, CD is 4 and cos(120) equals -(1/2). Thus, $AD^2 = 16 + 16 + (4)(4) = 48$. Hence, $AD = 4\sqrt{3}$ which is less then 7.

So while it is tempting to think that one can construct a tetrahedron from the "flat" triangles ABC and BDC in Figure 4 by flexing these two triangles along the line CB that they share so that A and D leave the plane of the triangle and a tetrahedron is formed, this will not work because this process will shorten the length of AD which is already too long at 7 units to fit! (If one tries to make a model of this "tetrahedron" using sticks, one can make something three dimensional only by having the 7 unit "strut" bend so that it can be forced into place with the sticks of length 4.)

So given 6 lengths is there some "easy" way to tell if there is a tetrahedron with these lengths, given their relative positions as in Figure 2b? The answer is "yes." Arthur Cayley and Karl Menger (with improvements by others) have shown how an answer can be determined by calculating a particular determinant, together with checking that the faces of the resulting potential tetrahedron would obey the strict triangle inequality. In essence these determinant calculations show that a tetrahedron exists by finding its volume, and as a fringe benefit of this approach, when the determinant is zero, it means that one has 4 points that lie in a single plane, and, thus, form a degenerate tetrahedron. Details can be found in Mussa [5] and Worth and Dreiding [6]. However, there are many natural questions that can studied by K-12 students to explore the "space" of 6-tuples that lead to tetrahedra. You can experiment by making models or using commercially available kits which allow one to make polyhedra with sticks of different lengths. It may be helpful to state a sufficient condition for the existence of a tetrahedron when the lengths for its edges are as shown in Figure 2b. We will say that a set of CT of positive integers is completely tetrahedral if, when any six lengths are selected from CT to use as labels in Figure 2b, one can actually construct the associated tetrahedron.

Theorem:

A set CT is completely tetrahedral if it is a subset of an interval $[t, t\sqrt{2}]$ containing at most one endpoint of this interval.

Remarkably, the smallest completely tetrahedral set of consecutive integers is 13, 14, 15, 16, 17, 18. Note that we are allowed to use numbers from this set more than once as edge length of our tetrahedra. However, using only the six numbers 7, 8, 9, 10, 11, 12 one can construct 30 incongruent tetrahedra where each of the numbers 7,...,12 appears!

Partition types of tetrahedra

Since a tetrahedron has 6 edges we will consider initially classifying tetrahedra by which partition of 6 the tetrahedron is associated with. For example, the partition {4, 2} is interpreted as meaning that one has a tetrahedron with two kinds of edges. There are four edges of the same length a and two edges of equal length b, but where the lengths a and b are different. We will use the letters a, b, c, d, e, and f in our diagrams to indicate the 6 different possible kinds of edge lengths, but often we will have edges of the same length. Also note that the sextuple associated with a particular diagram codes the values in the positions of the edges as in Figure 2b. Consider the following example of a {4, 2} tetrahedron - S(11,12,12, 11, 12,12). The diagram associated with this sextuple is shown in Figure 5.

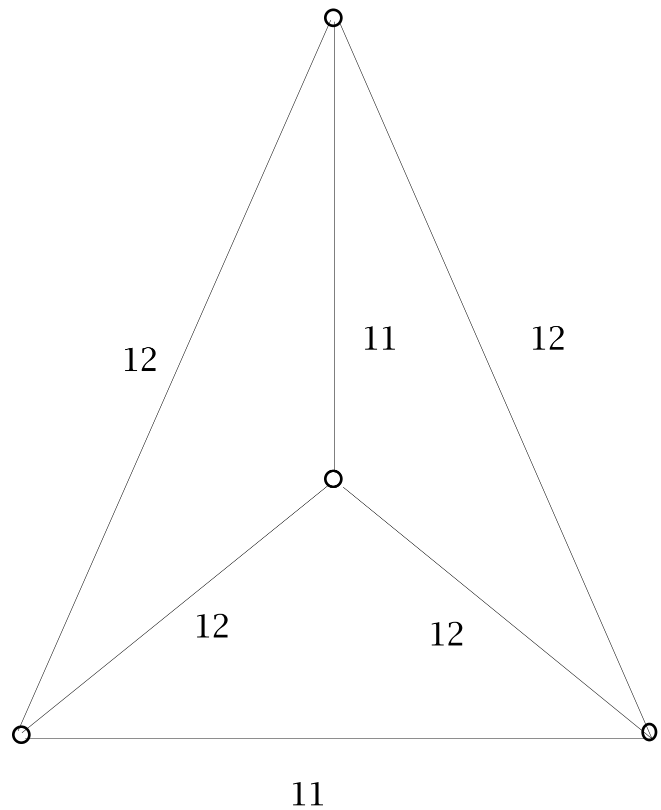

Figure 5 (An example of a {4, 2} tetrahedron)

Problem 4:

a. Verify that for every partition of 6 (it turns out there are 11 such partitions) there exists a tetrahedron with integer lengths which is associated with this partition.

b. Verify that for every partition of 6 one cannot draw 4 points not on a line in the plane and have integer lengths between the points (4 point determine 6 distances). Determine which partition types are possible for degenerate tetrahedra with and without the condition that the 6 lengths involved are integers.

While the fact that every tetrahedron can be associated with one of the 11 partitions of 6 is a nice fact, almost immediately one sees that there are other easy to spot properties of the pattern of edge lengths of tetrahedra that suggest even "richer" ways to classify tetrahedra. This is worthwhile because it encourages careful looking at geometrical phenomena as a way of making mathematical progress on problems. Look at the edge lengths in Figure 5. What do you notice? The edges of length 11 are disjoint from each other while the edges of length 12 form a circuit of length 4. Can we use simple properties of the pattern of the edges with the same edge length to get a better (finer) classification system for tetrahedra? When one looks at a graph with labels on the edges, the length of the edges here, it is convenient to think of these labels as colors. So is there a path of colored edges of length 2? Is there a set of matching edges of the same color? A matching in a graph is a collection of edges that don't have any vertex in common. For the tetrahedron if a collection of colored edges form a matching, it turns out there are exactly two edges in the matching. If one takes into account simple graph theory issues of this kind, on top of the use of partitions, how many different types of tetrahedra are there?

To get you started, think about the {4, 2}. The two edges thought of as having the same color can form either a matching or a path. These are only two possibilities. In the case of a path of length 2, the other four edges of the same length must form a circuit of length 3 with an additional edge at one of the vertices of the circuit. In the case of the matching, the other four edges form a circuit of length 4. Figure 6 shows the two cases side by side using the "lengths/colors" a and b. Note that we are not specifically interested in the relative size of a and b, so that we will consider the tetrahedron with four b edges forming a circuit with two a edges forming a matching and the tetrahedron having four a edges in a circuit with two b edges in a matching as having the same type.

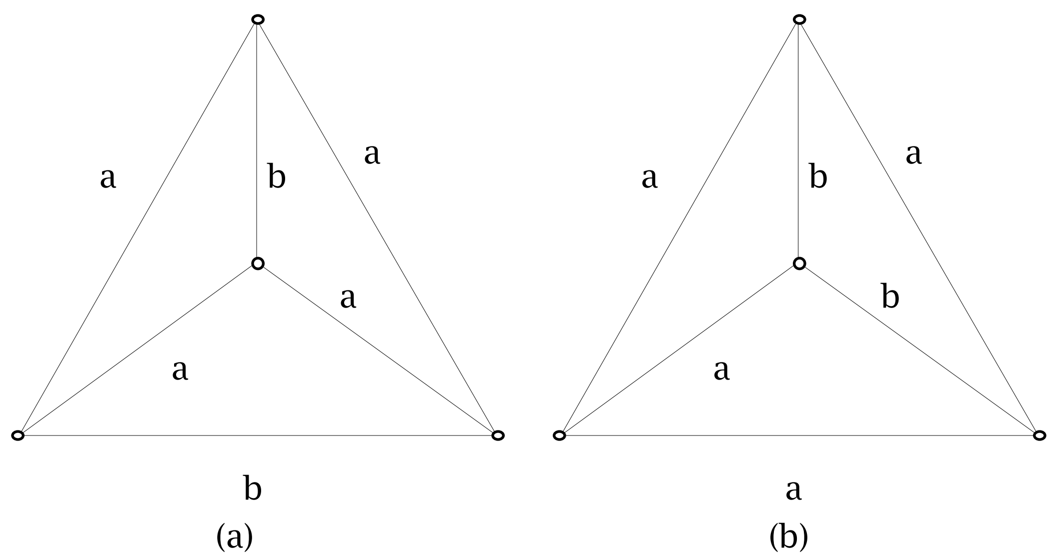

Figure 6

Figure 6 (Two different kinds of {4, 2} tetrahedra)

The sextuple associated with Figure 6a is S(b,a,a,b,a,a) while the sextuple associated with Figure 6b is S(a,a,b,b,a,a). Intuitively, it may seem that these two tetrahedra could not be congruent, but how can one be sure?

Here we can use some additional partition ideas that we mentioned earlier as well as seeing a way to help students broaden their view of geometrical phenomena. Consider the partition {6} where all of the edges of the tetrahedron have the same length. In this case we have a regular tetrahedron, all of whose faces are congruent, and even congruent equilateral triangles. For the type {6} we can write that its faces have congruence type {4}, by which we mean that all of the four faces are congruent. We can actually say a bit more. The faces have congruence type {4E}, meaning 4 equilateral triangles.

What partition types with respect to the congruence of faces can there be?

What is a complete list of partition types by faces if one takes the type of face, equilateral, isosceles, and scalene into account?

For example, we could have {4E}, {4I}, {4S}, and {3E, 1E}, {3E, 1I}, and {3E, 1S}, etc. However, just as we have proofs of interesting geometrical facts in plane Euclidean geometry, we can use simple geometry to prove that some of these types cannot exist. For example, {3E, 1E} would mean we have three congruent equilateral triangles, and one equilateral triangle which has a different edge length. This cannot happen. Though not "profound" we have the theorem:

Theorem: If three of the four triangles of a tetrahedron are congruent equilateral triangles, in fact, all four of the faces must be congruent equilateral triangles.

So, a student might ask what happens for the other types of triangles one might have:

Question: If three of the faces of a tetrahedron are congruent isosceles triangles, what can one say about the 4th face?

The answer to this question is rather interesting. If we have three faces of a tetrahedron with congruent isosceles faces, the fourth face can be an equilateral triangle or the fourth face can be an another isosceles triangle. In the first case we have a tetrahedron which is of partition type {4,2} while in the second case we have a {3,3} partition type tetrahedron!

Note that it might have been "nice" if we could tell tetrahedra apart based on congruence types, but because there are only 5 such partitions this turns out to be too much to hope for. Thus, the regular tetrahedron and the tetrahedra of the type corresponding to those in Figure 6a (from an edge point of view of type {4,2}) can't be told apart because they are both congruence type {4}, though with the refined use of the letters E, I, S we can tell them apart. They would be of types {4E} and {4I}.

It is fun for students and teachers alike to formulate questions of this kind and then to answer them! The important thing to note here is that geometry did not end with Euclid's Elements nor with a rich understanding of the classical non-Euclidean geometries. Even objects as seemingly simple as the tetrahedra offer many questions which are still not fully understood, not that hard to get some new results about, and just fun to explore.

For completeness, we show in Appendix I the 25 different partition types that one can classify tetrahedra into, together with the congruence type based on partitions of 4 augmented with the letters E, I, and S. One final question. Can one tell the 25 types apart using only partitions of 4 together with the labels E, I and S? (Hint: What about {1S, 1S, 1S, 1S}?)

References:

1. Alonso, Orlando. "Grünbaum's convex quadrangles enumeration and an Extension of the angle-side reciprocity of quadrangles." Geobinatorics 20 (2010) 45-47.

2. Alonso, Orlando B., and Joseph Malkevitch. "Classifying Triangles and Quadrilaterals." Mathematics Teacher 106.7 (2013): 541-548.

3. Alonso, Orlando B., and Joseph Malkevitch. "Enumeration via Partitions." Consortium 98 (2010): 17-21.

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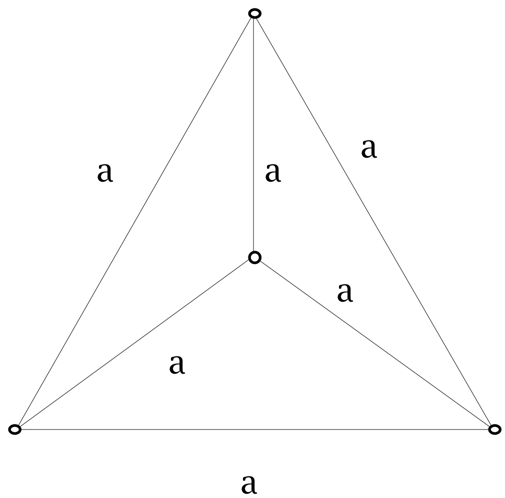
5. Mussa, Derege Haileselassie. Tetrahedra and Their Nets: Mathematical and Pedagogical Implications, Doctoral Dissertation, Columbia University, Teachers College, 2013 (Under the supervision of J. Malkevitch).

6. Wirth, Keith, and André Dreiding, "Edge lengths determining tetrahedrons." Elemente Math. 64 (2009) 160-170.

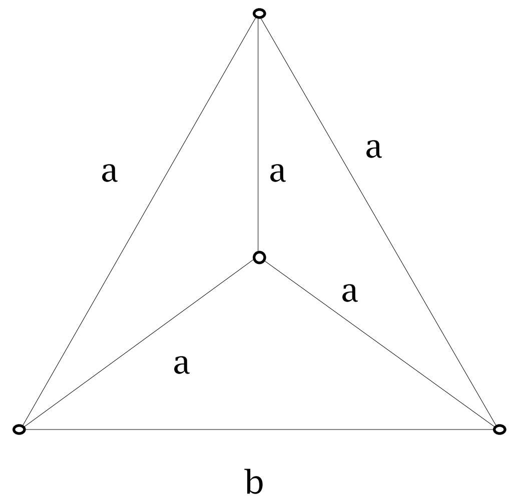
Appendix I

The 25 different types of tetahedra based on partitions of 6, the number of edges, and also indicating for each partition type the congruence of faces type, taking into account if the faces are equilateral (E), isosceles (I), or scalene (S).

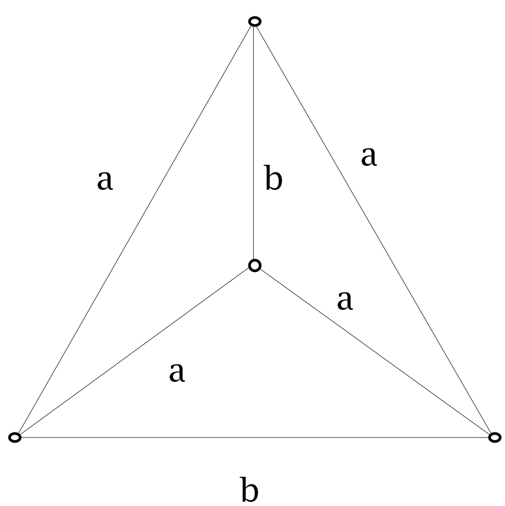
{6} and {4E}



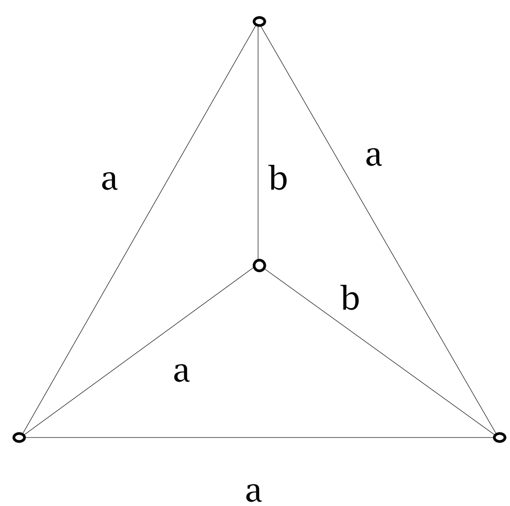
{5,1} and {2E, 2I}



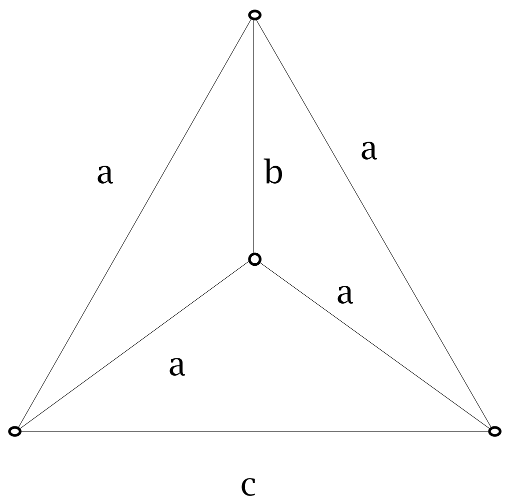
{4,2} and {4I}



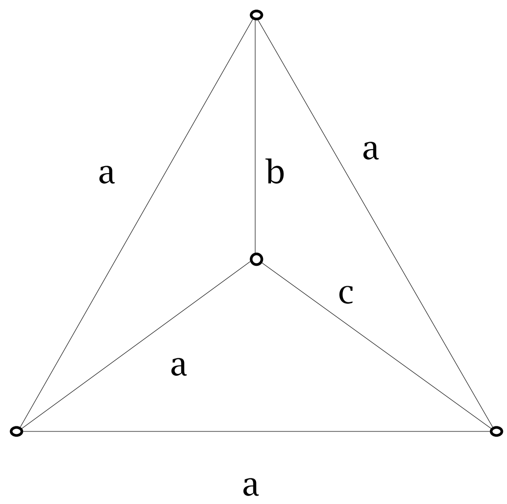
{4, 2} and {2I, 1E, 1I}



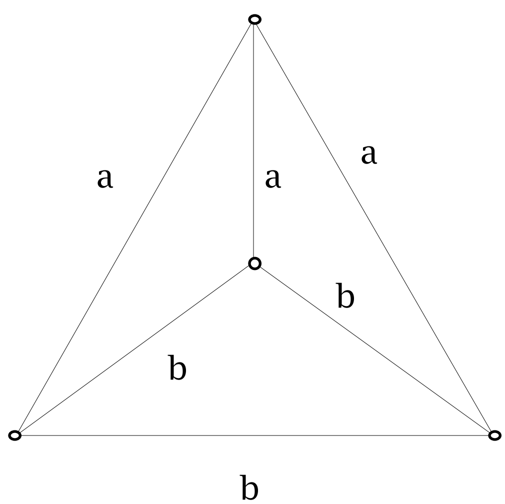
{4, 1, 1} and {2I, 2I}



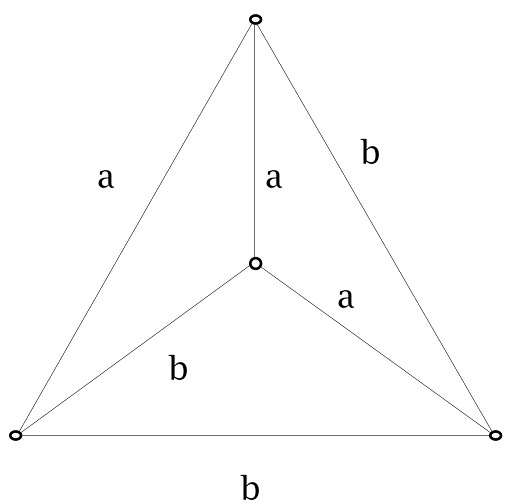
{4, 1, 1} and {1E, 1I, 1I, 1S}



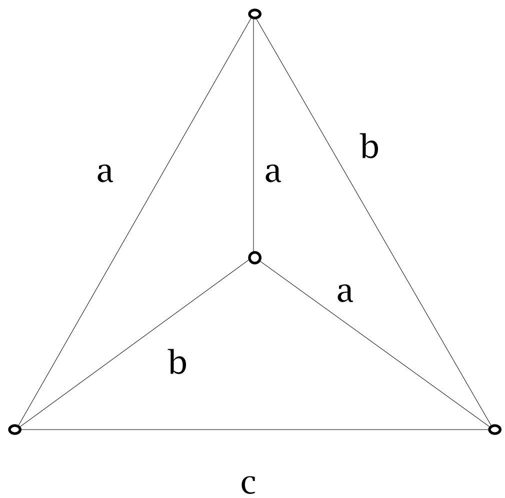
{3, 3} and {3I, 1E}



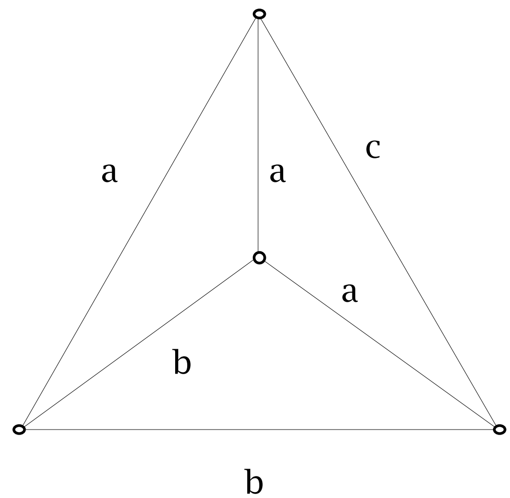
{3, 3} and {2I, 2I}



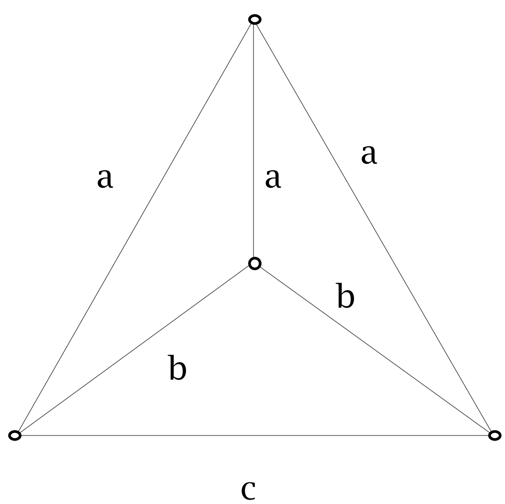
{3, 2, 1} and {2I, 2S}



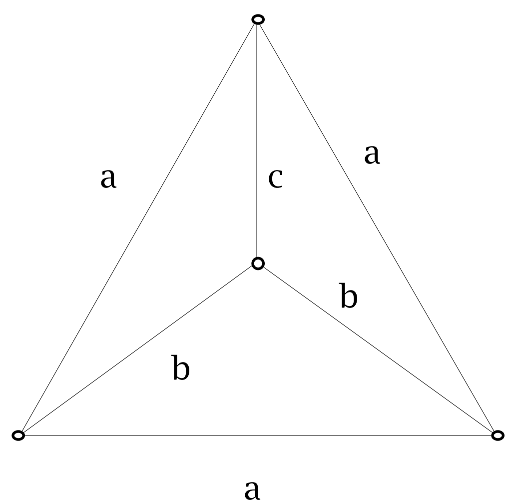
{3, 2, 1} and {1I, 1I, 1I, 1S}



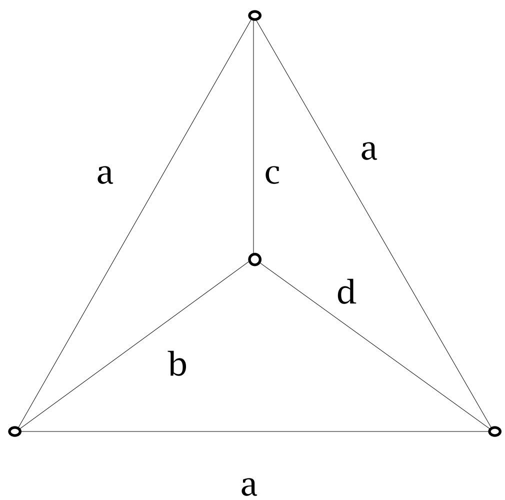
{3, 2, 1} and {2I, 1I, 1I}



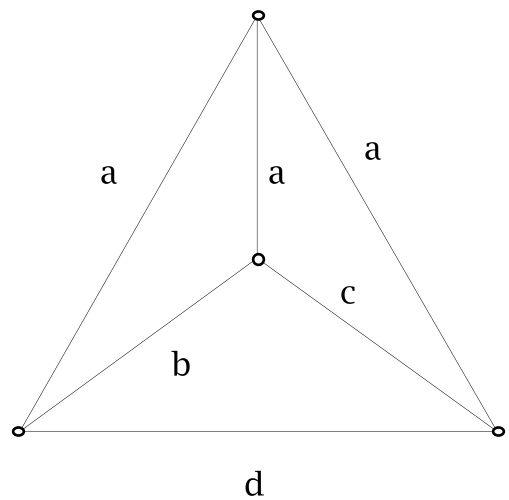
{3, 2, 1} and {2S, 1E, 1I}



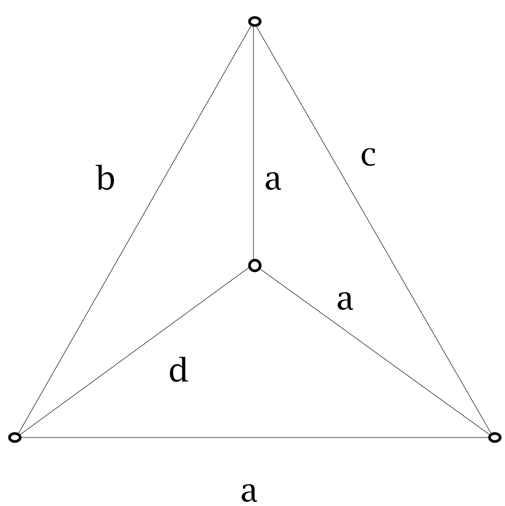
{3, 1, 1, 1} and {1E, 1S, 1S, 1S}



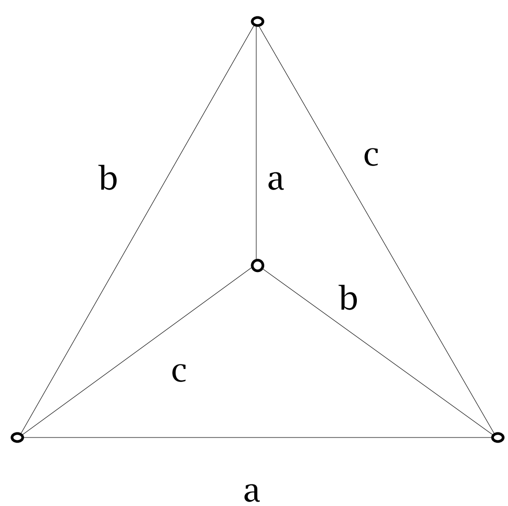
{3, 1, 1, 1} and {1I, 1I, 1I, 1S}



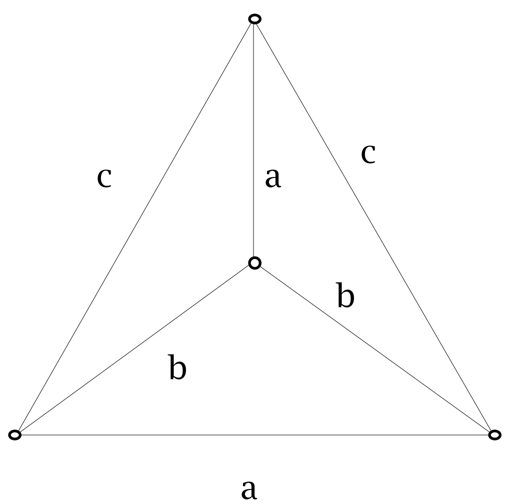
{3, 1, 1, 1} and {1I, 1I, 1S, 1S}



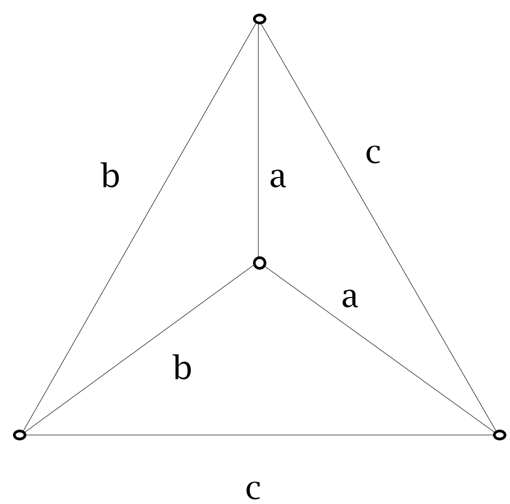
{2, 2, 2} and {4S}



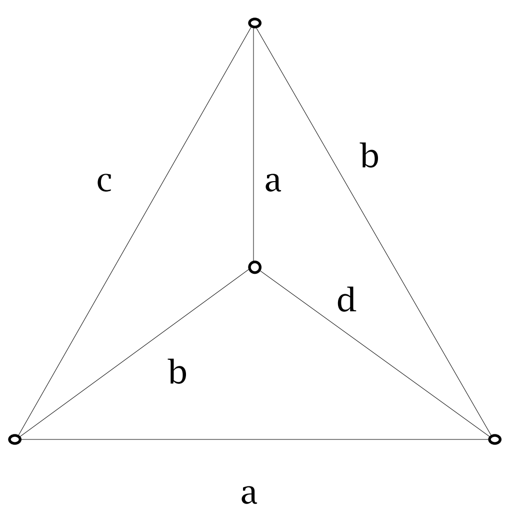
{2, 2, 2} and {2S, 1I, 1I}



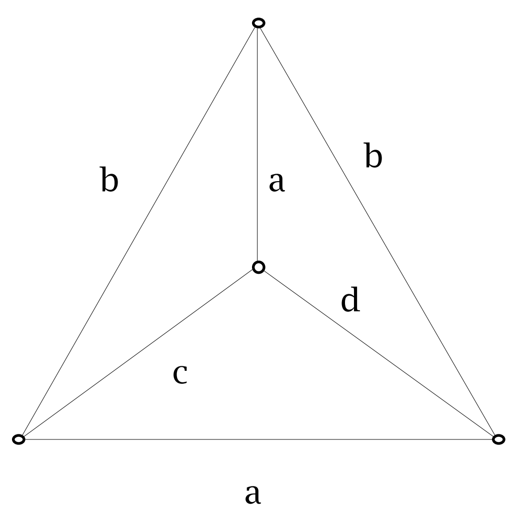
{2, 2, 2} and {1I, 1I, 1I, 1S}



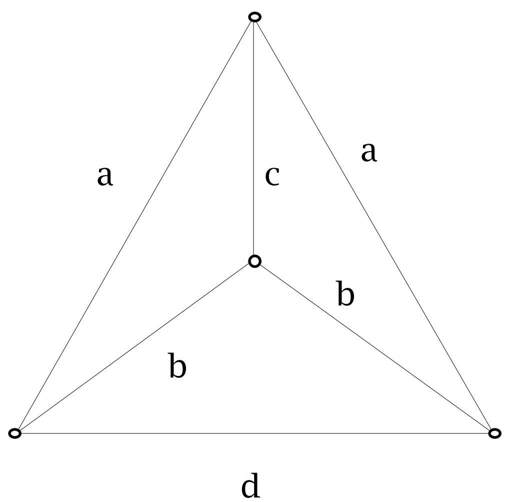
{2, 2, 1, 1} and {2S, 2S}



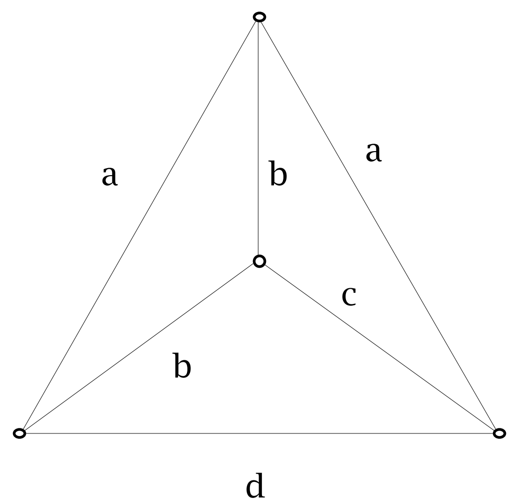
{2, 2, 1, 1} and {1I, 1S, 1S, 1S}



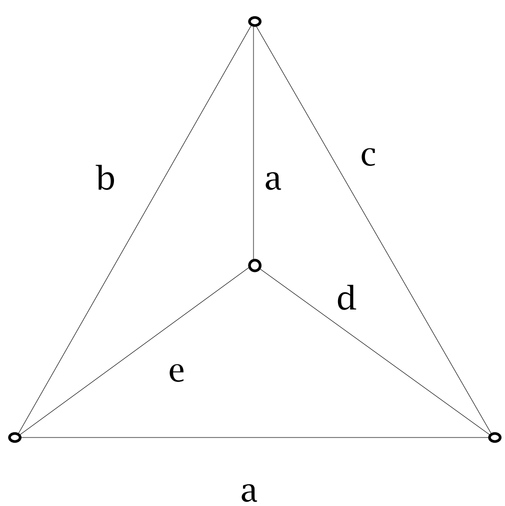
{2, 2, 1, 1} and {2S, 1I, 1I}



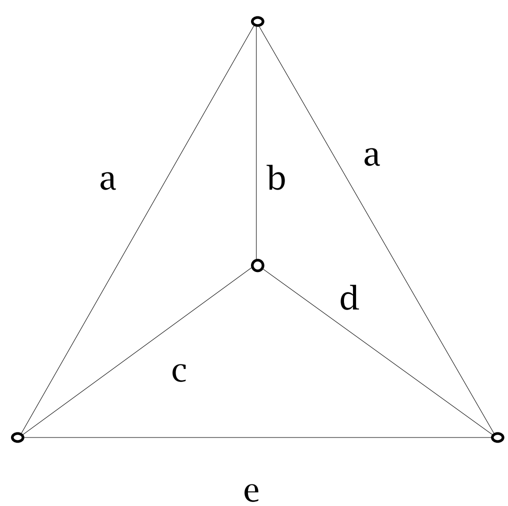
{2, 2, 1, 1} and {1I, 1I, 1S, 1S}



{2, 1, 1, 1, 1} and {1S, 1S, 1S, 1S)



{2, 1, 1, 1, 1} and {1I, 1S, 1S, 1S}



{1, 1, 1, 1, 1, 1} and {1S, 1S, 1S, 1S}