Mathematics and CS Club

Planar graphs and knots

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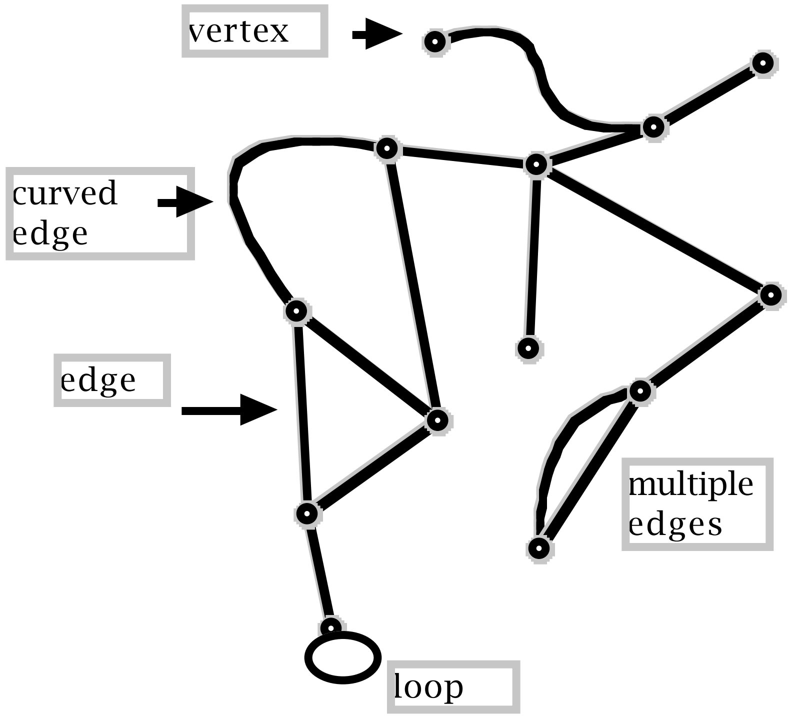
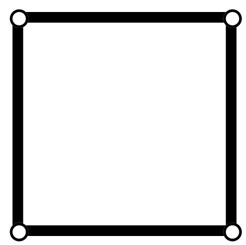
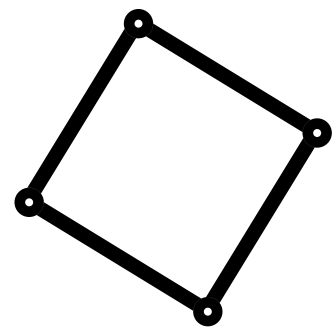
Graph: Dots and curves (some straight) diagram



Are these graphs the same or different?





As graphs, they have the same structure so are called "*isomorphic*."

Isomorphic (same structure) disconnected graph

This graph has two pieces or components. One piece graphs are called connected. "Crossing" not a vertex.

Isomorphic (same structure) connected graph

Two schools of graph theorists:

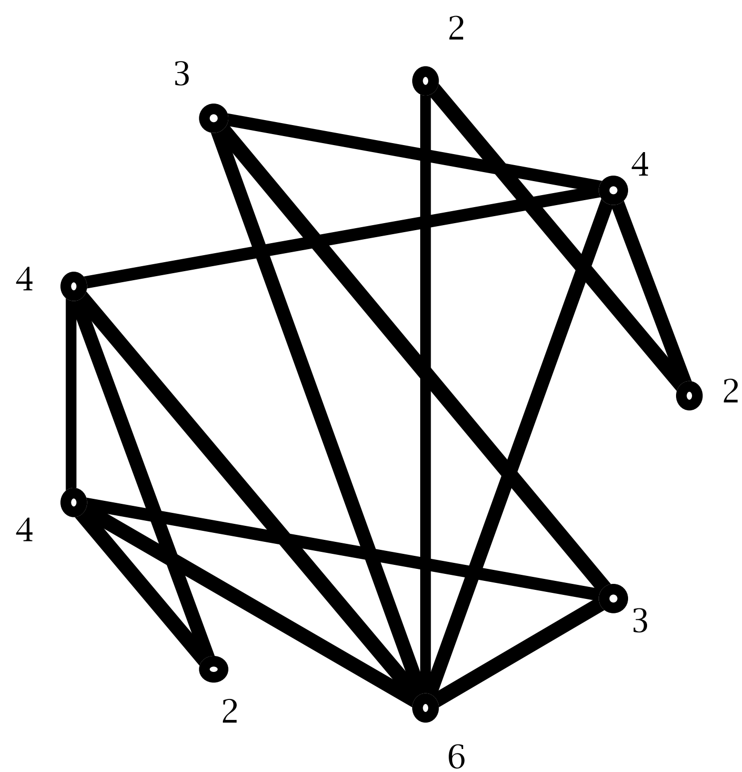
a. Graphs never allow loops or multiple edges

b. Graphs can have loops or multiple edges

One talks about simple graphs in b., to get to case a.

When a graph is connected one can "move" along edges between any pair of the vertices of the graph.

The number of edges at a vertex is called the *degree* or the valence of the vertex. Each vertex is labeled with its degree. There are three 4-valent vertices.



Each edge connects exactly two distinct vertices, so adding the valences of all the vertices gives:

Theorem: (Leonhard Euler)

The sum of the valences of a graph (no loops) is twice the number of edges.

Corollary: A graph has an even-number of odd-valent vertices.

Graphs are an interesting topic for theoreticians in mathematics and computer science, but also are of value in applications that non-mathematicians as well as mathematicians can put to use:

Vertices represent "real world things"

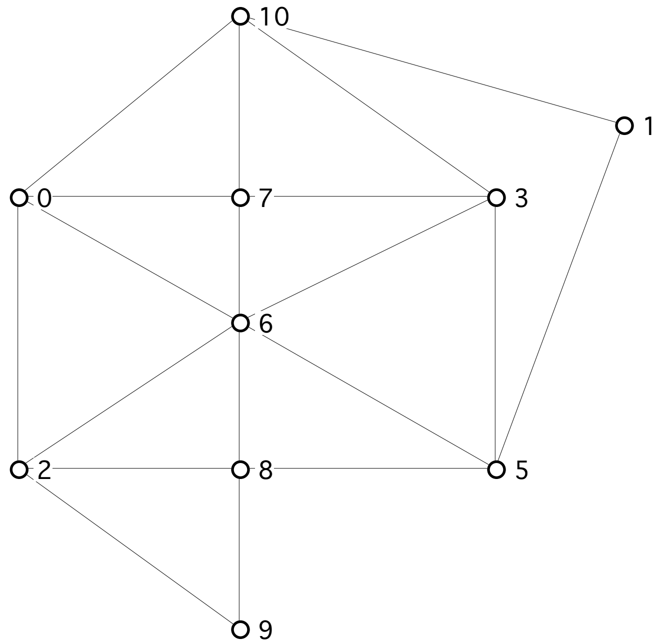
Edge: join to vertices with an edge when the two objects represent obey some relationship

Vertices represent: York College students; Colors; buildings in Jamaica, etc.

Example using this tool in Operations Research:

For the collection of roads shown, is there a **snow removal** tour for a plow starting at vertex 0 and which returns to 0 having traversed each edge once and only once?

(Find efficient snow removal route)



Graph H

Note that when such a tour is possible it does not matter what the weights of the edges might be - times or distances.

However, for most diagrams of this kind such a tour is not possible so one seeks a "tour" visiting each edge at least once, returning to where it started, and with a minimum number of repeated edges!

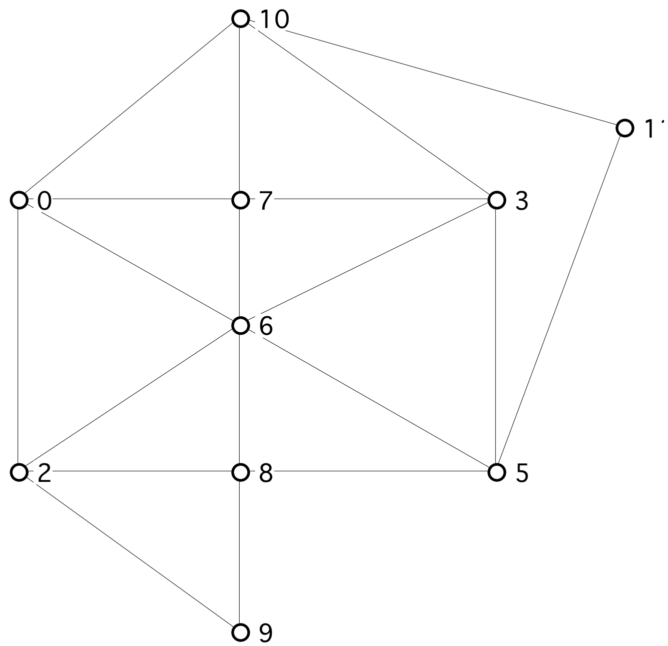
This problem is known as the *Chinese Postman Problem*, honoring Meigu Guan who in the 1960's pioneered its study!

(Polynomial time algorithms are known for solving it.)

A connected graph has an *Eulerian circuit* if and only if all of its vertices have even degree (even valence).

Early work in graph theorem can be tied to work that Leonhard Euler did about 1735. Hence, Euler is honored because he "pioneered" the study and application of Euler circuits.

Take another look at at Graph H:

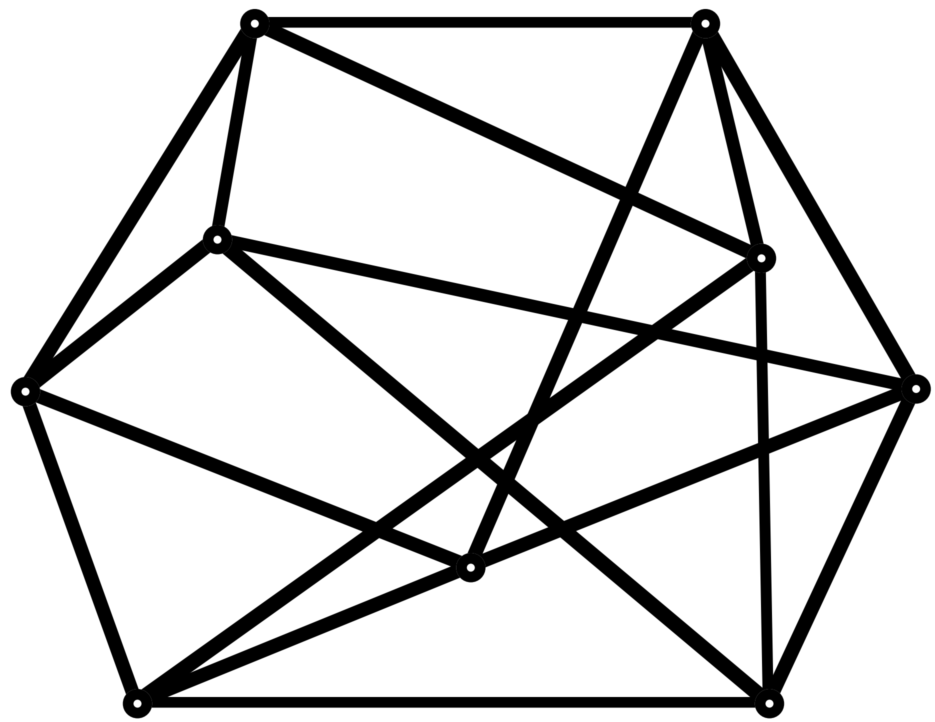


Graph H

The graph H above has the special property that it has been drawn in the plane so that if edges meet they meet only at vertices. Is it true that every graph has a drawing in the plane?

Such graphs are called *planar*; when drawn in the *Euclidean plane* in this way, they are called *plane* graphs and are "embedded" in the plane.

Perhaps this graph can be redrawn to avoid the 8 crossing that occur?



What drawing has the minimum number of crossings?
This number is called the crossing number!

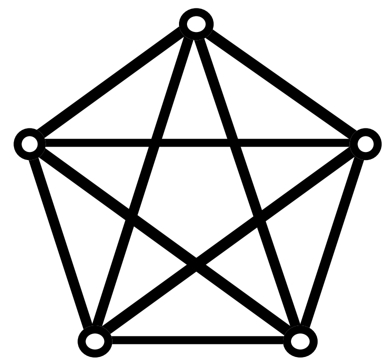
Fact: Not every graph is planar.

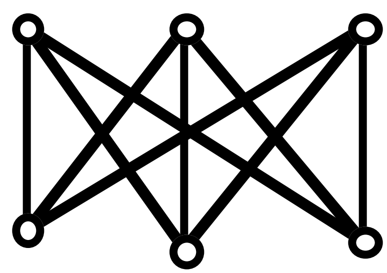
Kuratowski's Theorem

Kazimierz Kuratowski (1896-1980)

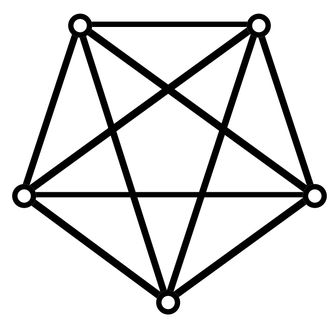
Theorem proved about 1930, paper in French.

Any non-planar graph "contains" a copy of one or both of these two graphs:





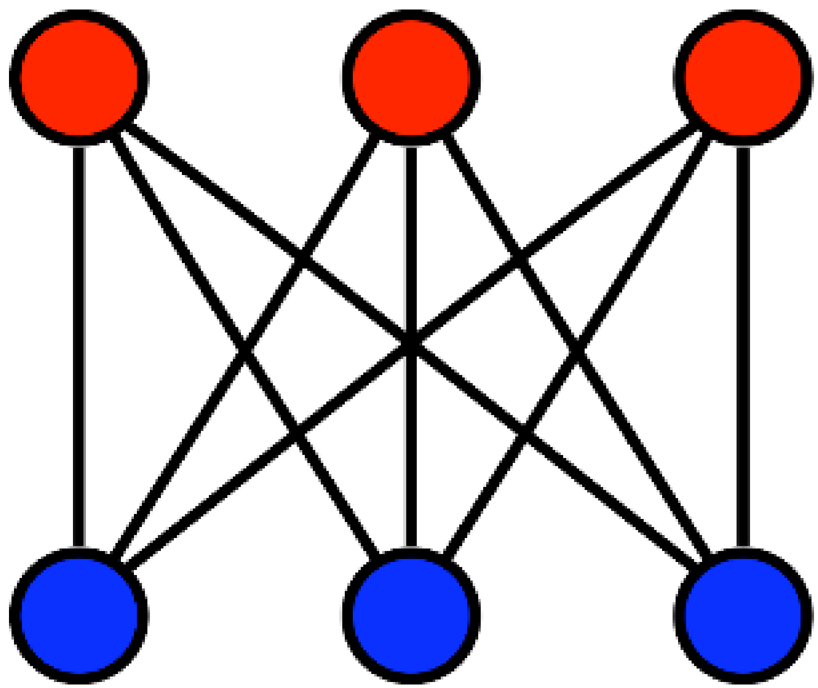
5 vertices each joined to every other - complete graph on 5 vertices



$K_{3,3}$ Two sets A and B of three points each joined to all of the others but no edges between points of A or points of B. Complete *bipartite* graph with sets of size 3.

Can one connect up 3 houses and 3 utilities so that the connections as drawn in the plane meet only are vertices?

Answer: No!! (Associated graph is non-planar)

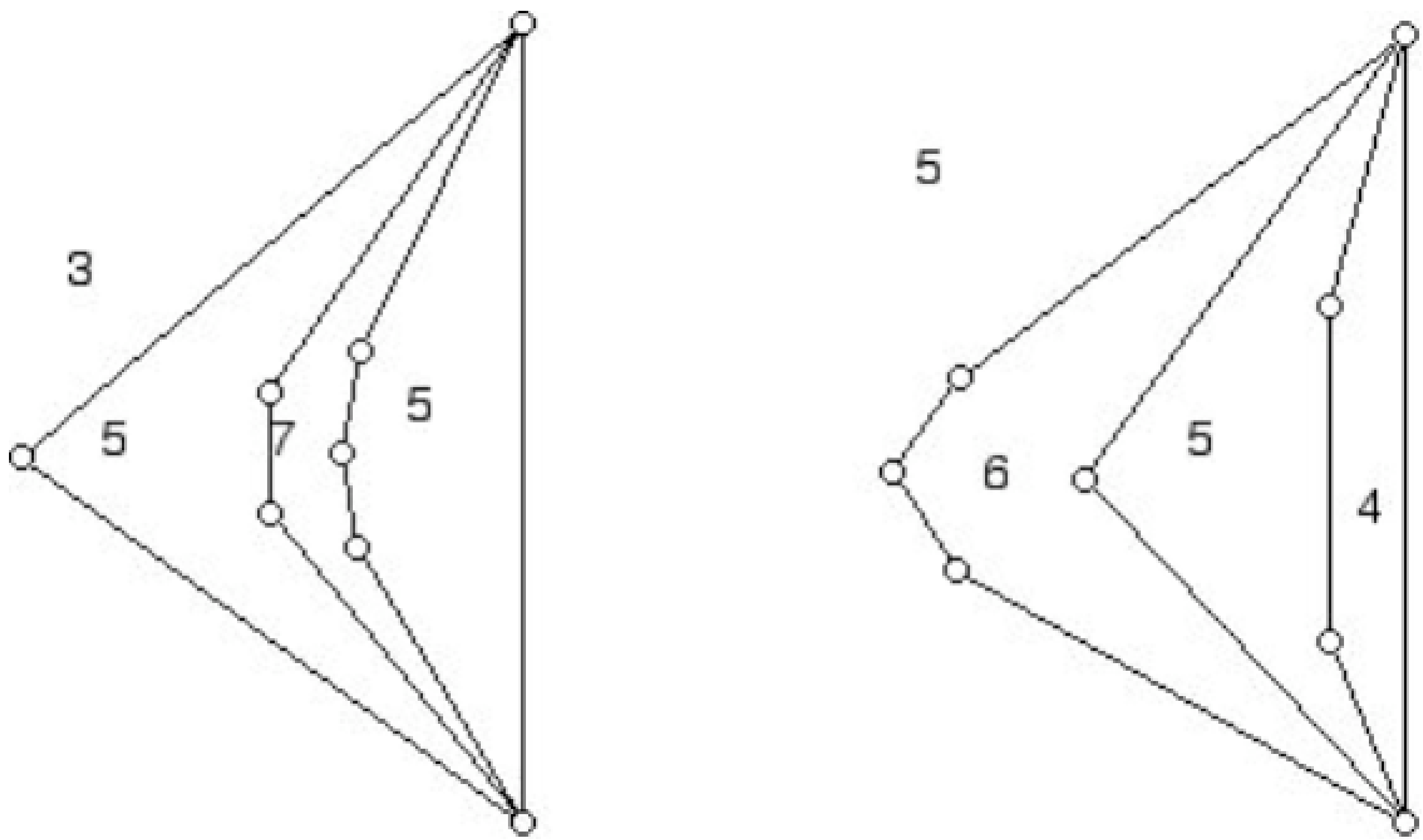


Computer science problem:

Develop an algorithm which "quickly" tells if a graph has a plane drawing, and if so, creates such a drawing!

When planar graphs are drawn as plane graphs in the plane, they not only have vertices and edges, but regions or faces - the number of sides of each such region can be counted!

Count the number edges an "ant" encounters as it walks around a region and returns to where it began!!



Perhaps surprisingly, two isomorphic graphs, as above, can be be drawn in the plane with different sizes for the faces that it determines.

Graph theory is not only a very applied topic but gives insights into topology - "rubber sheets" geometry.

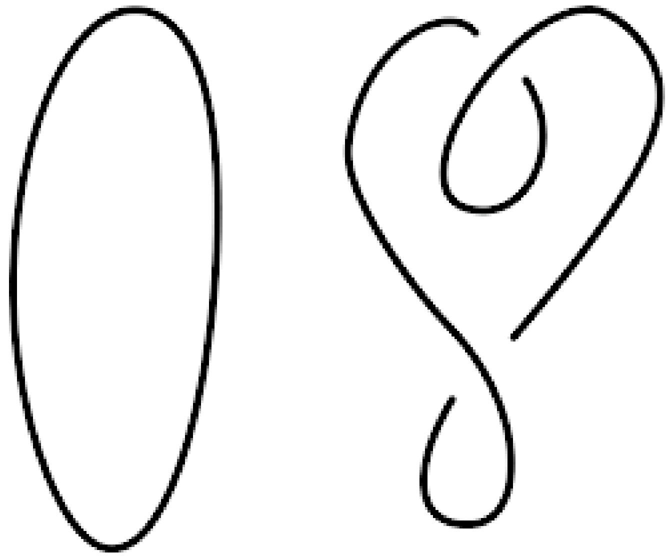
In particular it helps with understanding *knots*.

What is a knot?

Intuitively take a piece of string with two ends and after some perhaps convoluted operations with the ends of the string fuse (join) its endpoints.

The simplest form of a knot is shown below, with the point where the two ends are fused is not highlighted.

Knots "live" in 3-space but can be represented in two dimensions by projecting down the knot into a plane - "crushing down" the knot "flat." The result is typically a 4-valent graph.



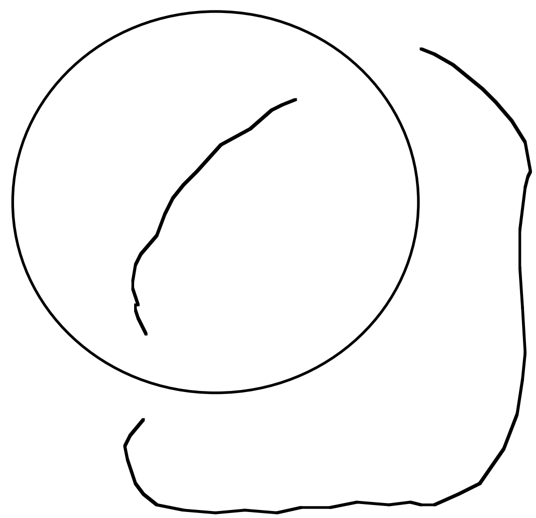
(Image from Wikipedia)

Note: One can "code" in such "knot drawings" what strand goes *under* or *over* another "strand" of the knot.

These two knots are "isomorphic."

But this piece of string does not appear knotted, and in fact, it is the "special knot" called the unknot!

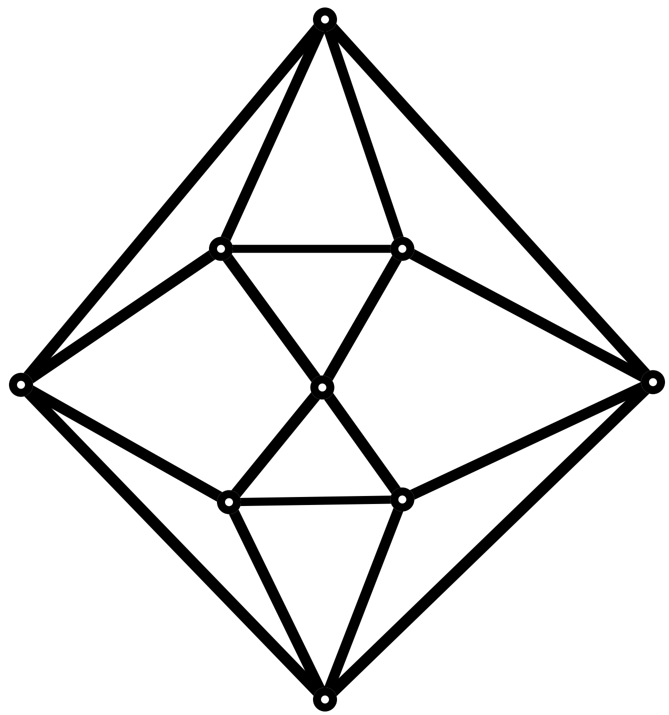
When two pieces of string each separately fused interact, the result is called a "*link*." Here is a two component link, and the pieces can be pulled apart.



Some links cannot have the "strands" separated!

Recently knot theory has found many applications in biology, in particular in the study of DNA and the structure of proteins.

Drawings of knots can be interpreted as plane 4-valent graphs (perhaps with multiple edges) by interpreting the crossings as 4-valent vertices, and suppressing vertices of valence 2 that might appear.



Note: The diagram above is 4-valent and hence has an Eulerian circuit!

It has a SPECIAL KIND of Eulerian circuit, known as a "***cut-through***" Eulerian circuit. Pick any. Move along it in either direction and when one gets to the "next" vertex, pick the edge in the middle (rather than the edge on the "far left" or "far right."

Cut-through path at a 4-valent vertex of a plane graph:

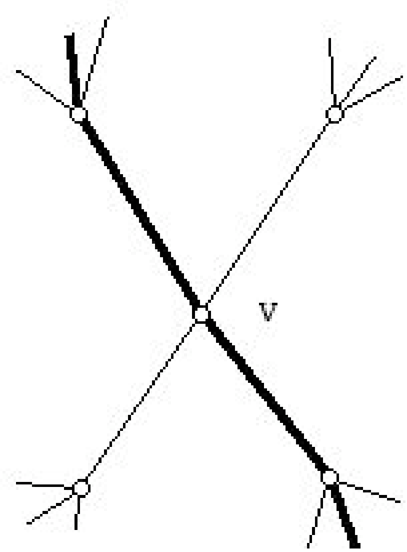
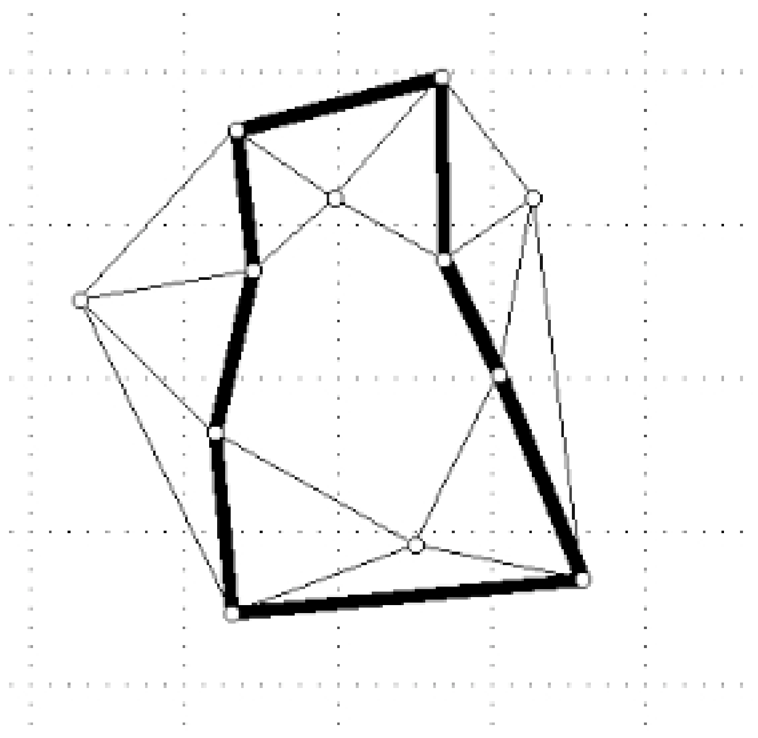
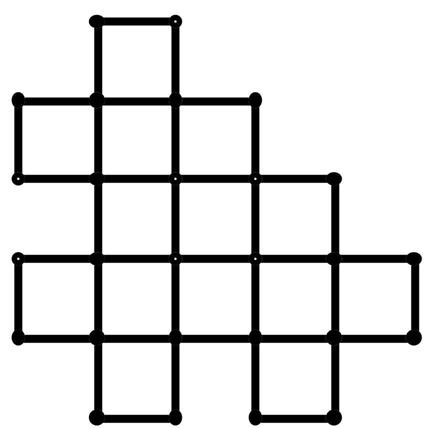


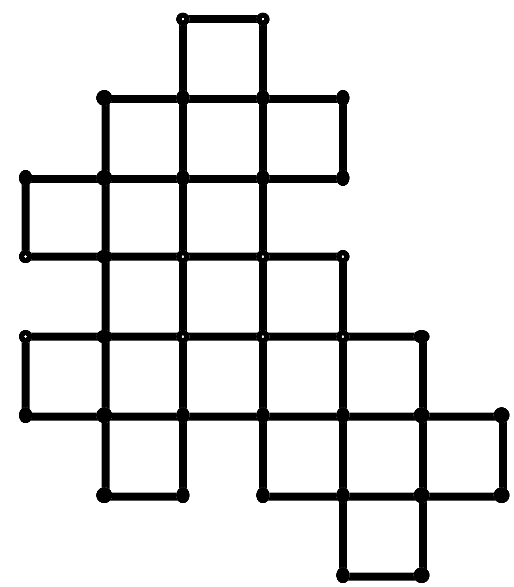
Diagram not the projection of a knot. Can can be thought of as a link with three pieces.



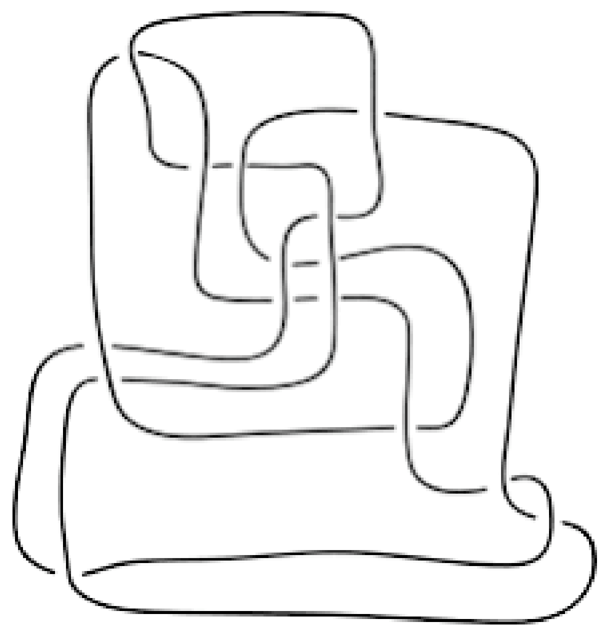
Polyominoes connected to knot theory:

(clusters of 1x1 squares, each that meet along edges and without "holes." Below: on left, a link of three pieces and on the right a polyomino that can be interpreted as a knot! (suppress the 2-valent vertices)

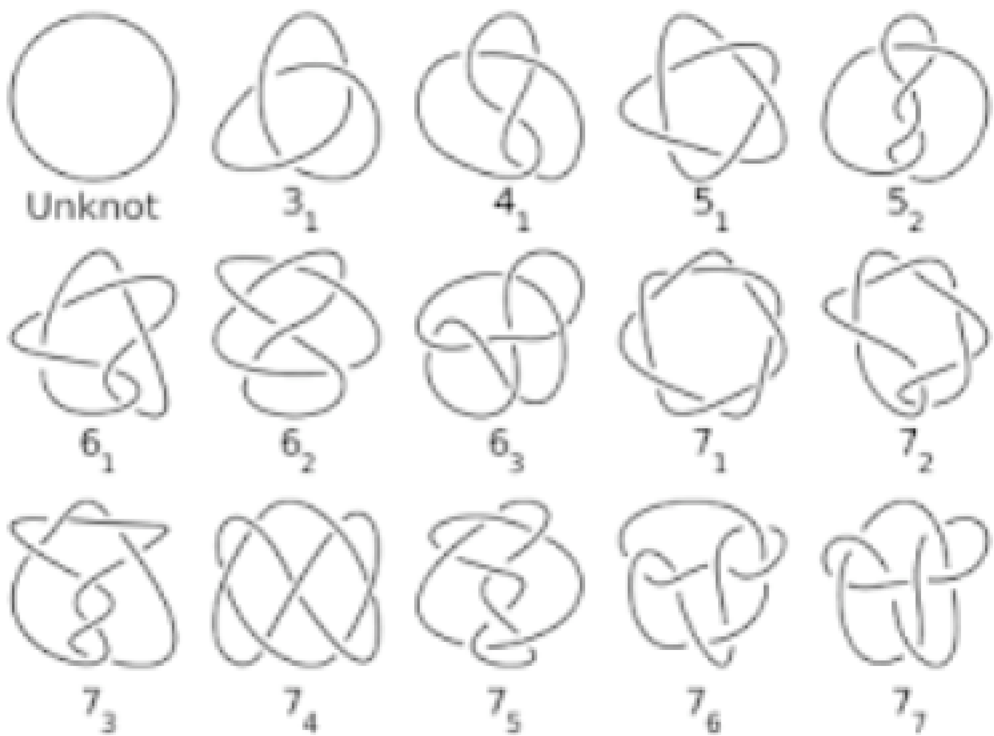




Hard questions: When can one simplify a knot drawing to the unknot? How many inequivalent knots are there with exactly k crossings when thought of as a planar graph?



Very small table of knots: Organized by the number of crossings:



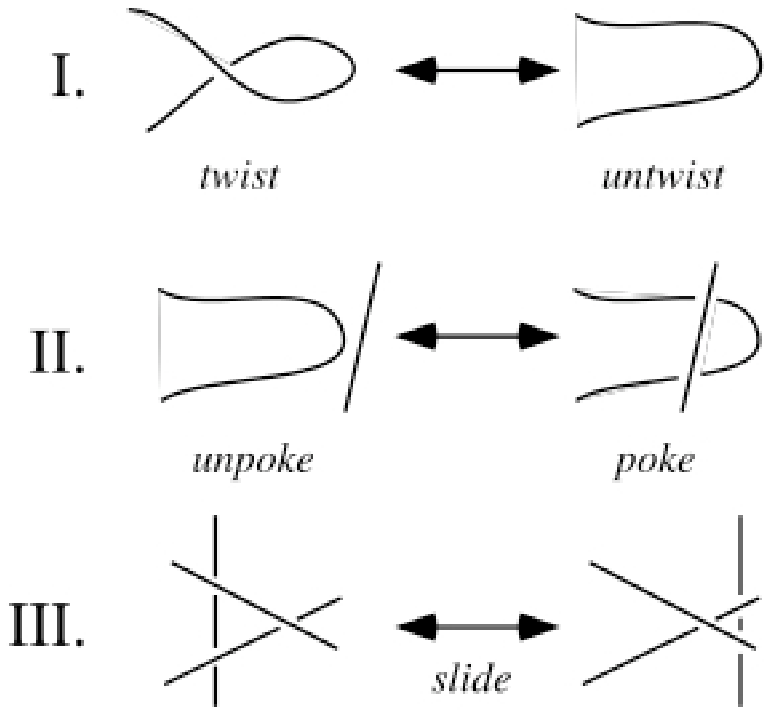
It turns out one can use graph theory to help answer this question.

The key idea was developed by the mathematician Kurt Reidemeister (1893-1971)

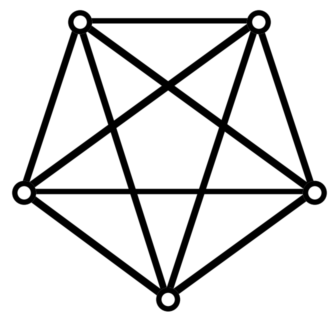
If two knots are "equivalent" one can be transformed to the other by a sequence of the three Reidemeister moves!!

Fact: During this transformation the number of crossings may go UP before it goes back down again. In particular, if the original drawing of the knot is REALLY the "unknot."

(Image from Wolfram Mathworld):

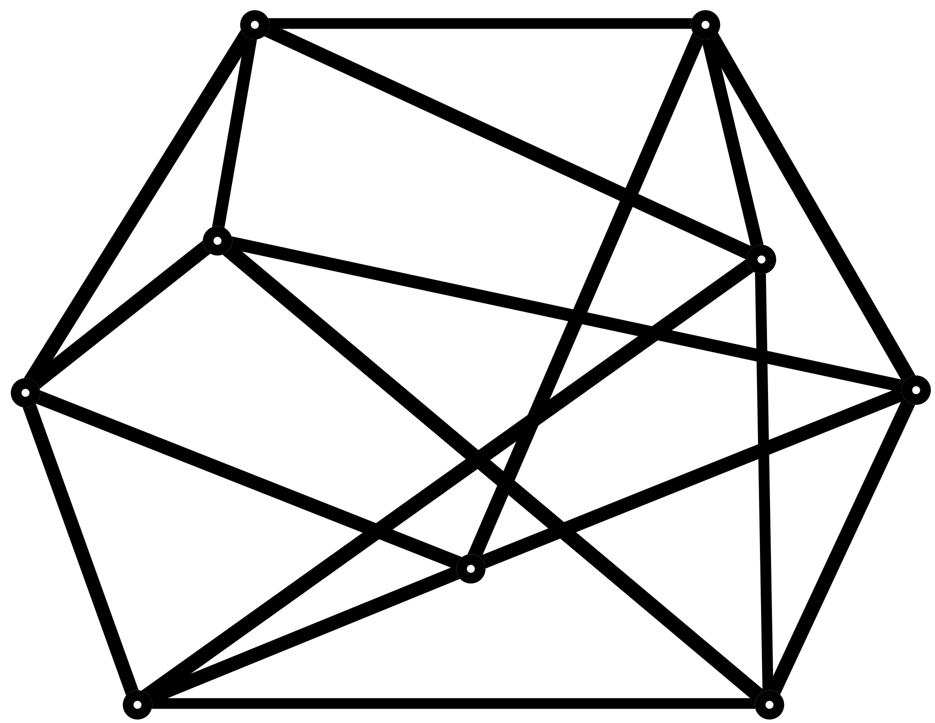


"Exercise" This graph can be thought of as a family of knots by making decisions about under and over information at each "cut-through" vertex. What different knots can you get from this single knot?



The graph below has 8 vertices and is drawn in the plane but this drawing is not plane.

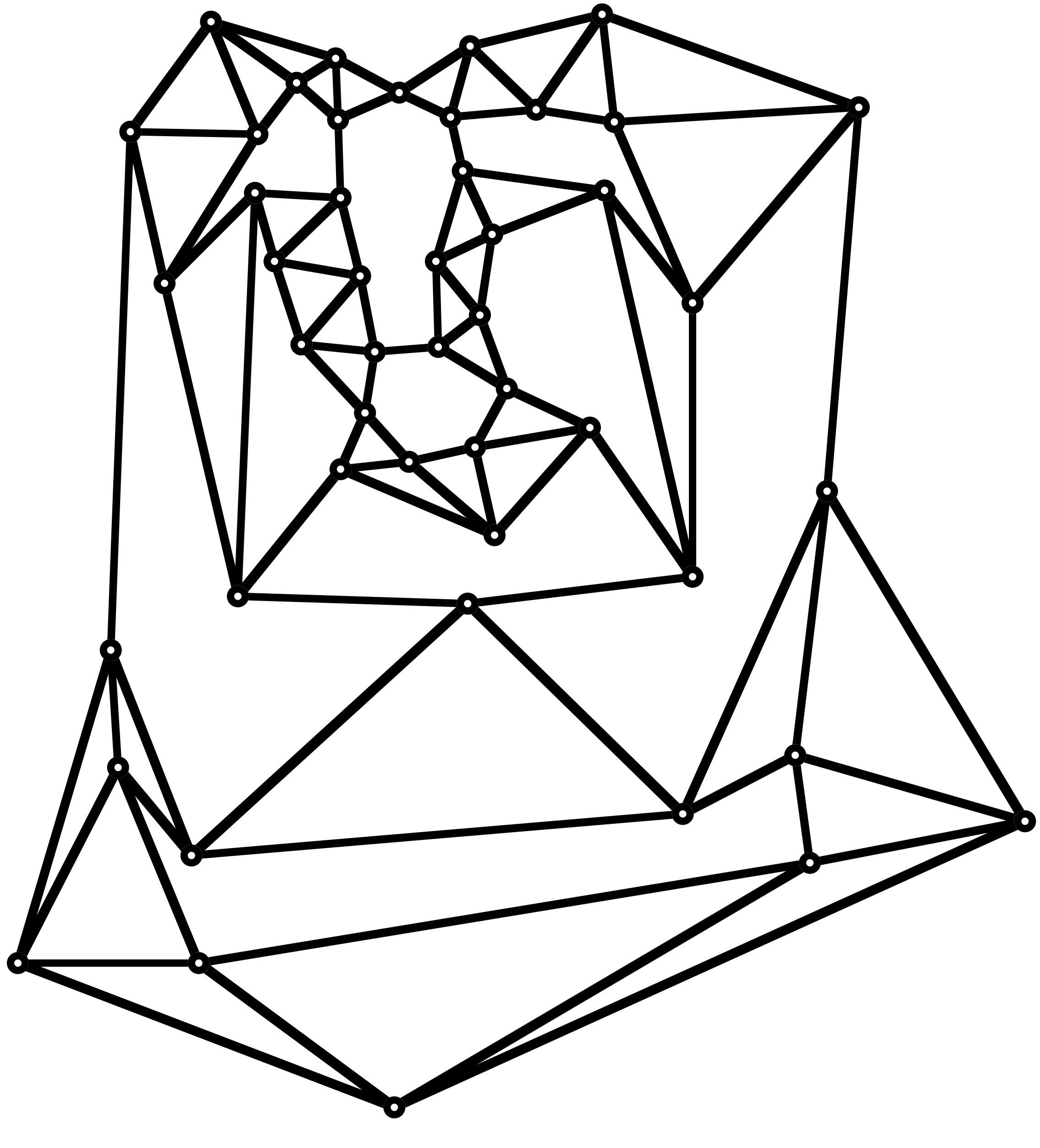
This graph has a Euler circuit, and it also has an Euler circuit when the "accidental crossings" are made into vertices.



One can classify 4-valent plane graphs according as the number of sides of the faces it has.

Example below:

4-valent plane; regions are all 3-gons and faces with an even number of sides. The "interior" faces are triangles or 6-gons and there is one face, the infinite face, that has 12 sides.



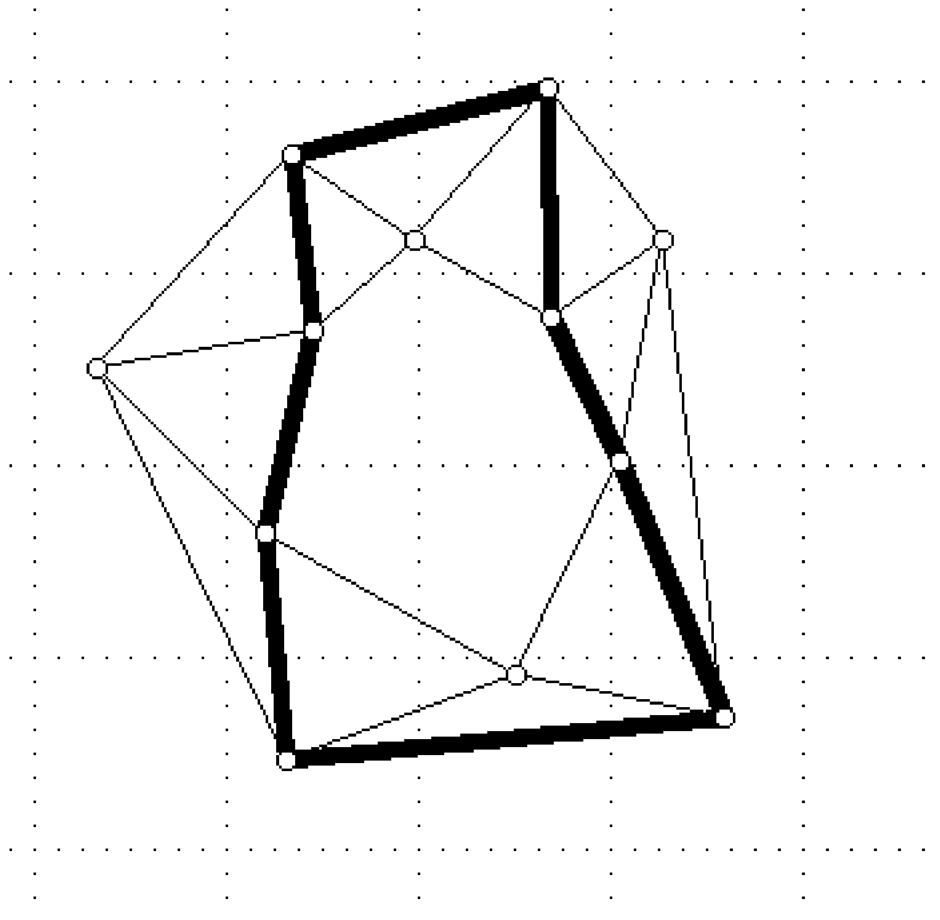
Theorem (1967):

A 4-valent connected graph, all of whose faces have a number of sides which is a multiple of 3 (e.g. 3-gons, 6-gons, 12-gons, etc. - called a *multi-3-gon*) can **never** be the projection of a knot.

(If one allows some number of 4-gons, one can find a knot with these parameters. as shown (1990) by Dalyoung Jeong (Seoul, Korea.)

Open questions:

a. When are all the cut-through circuit lengths in a plane 4-valent multi-3-gon graph ALL of the same length.



6-gonal anti-prism

b. Given a set of (suitable) positive integer lengths, when is there a plane 4-valent multi-3-gon graph with cut-through circuits of the these given lengths?

Comment: Many uninvestigated problems in discrete geometry are such that it is easy to get some partial results. Often what is hard is to show that all the "possibilities" are understood.

It is very EXCITING to make progress of problems that are not yet understood.

Sometimes there are not a lot of people interested in these problems. Much easier to do new "theoretical" mathematics than to do mathematics that has "clear applications." Sometimes the theory of today (think Euler in 1736) becomes the applied mathematics of tomorrow (Guan, Chinese Postman Problem, 1960).

Thanks for your attention!

Comments?

Questions?

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