

Convex 3-polytopes whose faces are equilateral or isosceles triangles

Joseph Malkevitch (Emeritus)

Department of Mathematics
York College (CUNY) (and CUNY Graduate Center)

email

jmalkevitch@york.cuny.edu

web page

<https://york.cuny.edu/~malk>

Dedicated to the memory of:

Richard Pollack

Branko Grünbaum

Two great late 20th century and
early 21st century discrete
geometers!

Historical context:

Convex 3-dimensional (polytopes) solids with "regularity" conditions:

a. (Euclid, fl. 300 BC) Five regular polyhedra

Congruent regular convex polygons;
same number of polygons at each vertex.

Euclid's *Elements* discusses the straightedge and compass construction of these five polyhedra commonly known as the *Platonic Solids*.

b. (Archimedes c. 287 BC to c. 212 BC) 13 Archimedean convex polyhedra (solids)

Archimedes's manuscript has been lost; modern knowledge from Pappus (c. 290 - c. 350); discussed by Kepler (1571-1630).

Regular polygons, same pattern at each vertex.

13 using the modern "group theory" definition that any vertex can be mapped by an isometry to any other (vertex transitivity)

However, identical patterns of faces around each vertex, there are 14 convex examples.

Archimedes made a
mistake and Pappus and
Kepler did not correct
him!

*Modern lesson: Proving an
enumeration is correct is not so easy!*

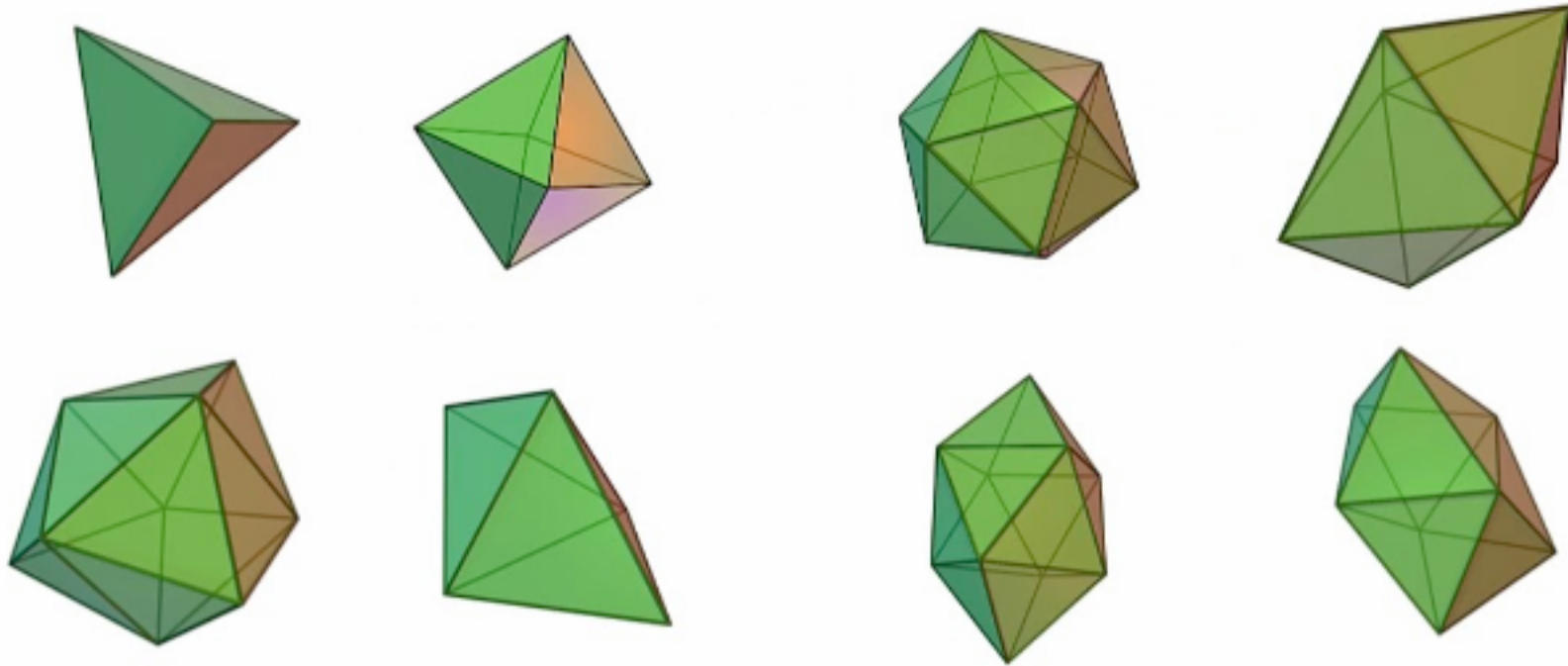
c. Convex

Deltahedra

(Equilateral
triangle faces)

Theorem:

There are exactly 8 convex polyhedra all of whose faces are equilateral triangles.



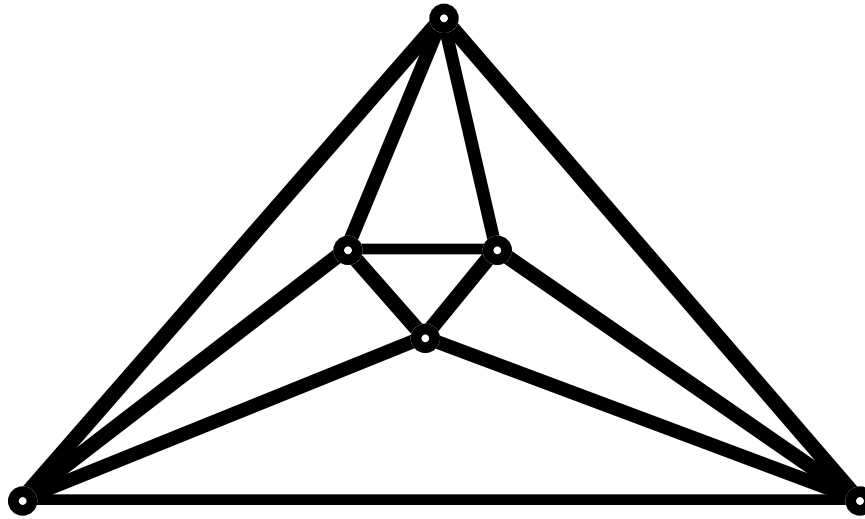
Hans Schepker (glass artist)



Glass convex deltahedra that I "commissioned."

Unfortunately, not all the same edge length. They make interesting shadow patterns of the edges as they are moved around.

A plane triangulation:

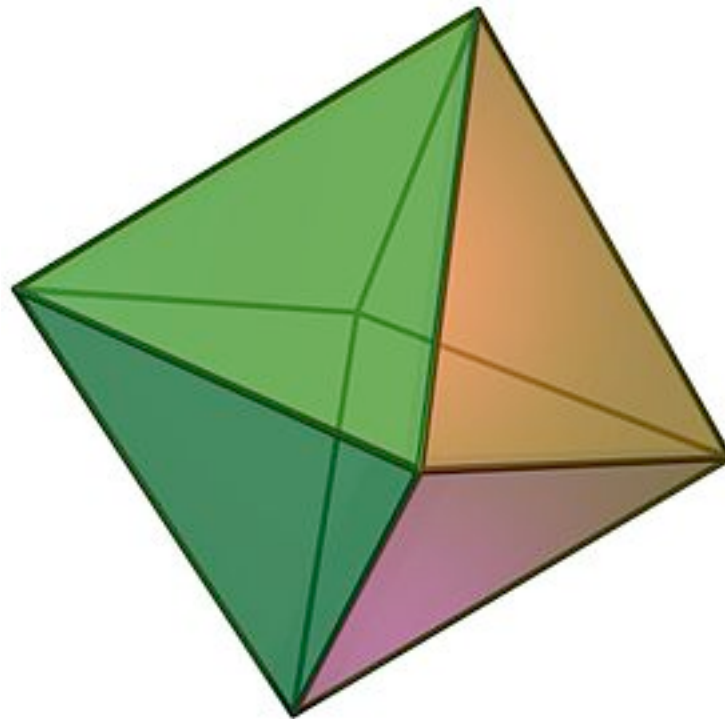


6 vertices; 12 edges

8 faces (each a triangle, including the unbounded or infinite face, which has 3 sides)

(Combinatorially, the regular octahedron)

Hidden line representation of the 3-dimensional convex polyhedra which realizes the previous graph:



Theorem: (Ernst Steinitz, 1922; Branko Grünbaum and Theodore Motzkin, 1963)

A graph is isomorphic to the vertex-edge graph of a convex 3-dimensional polyhedron (e.g. 3-polytopal) if and only if G is planar and 3-connected.

Planar: Isomorphic to a graph drawn in the plane where edges meet (intersect) only at vertices.

3-connected: For any pair of vertices u and v there are at *least* three simple paths from u to v that have only u and v in common.

Plane triangulations obey:

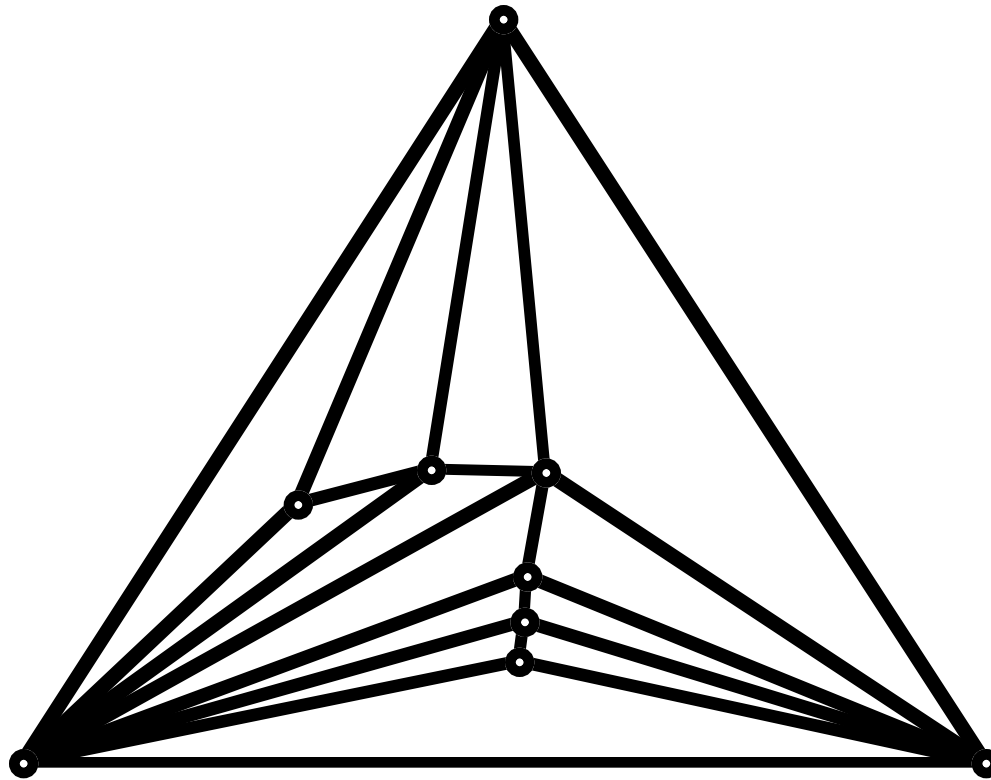
$V + F - E = 2$ (Euler's polyhedral formula)

and

$3F = 2E$ (F must be even, and the number of edges must be divisible by 3.)

For some situations below one needs 4 divides F or that E be even.

Less symmetrical triangulation:



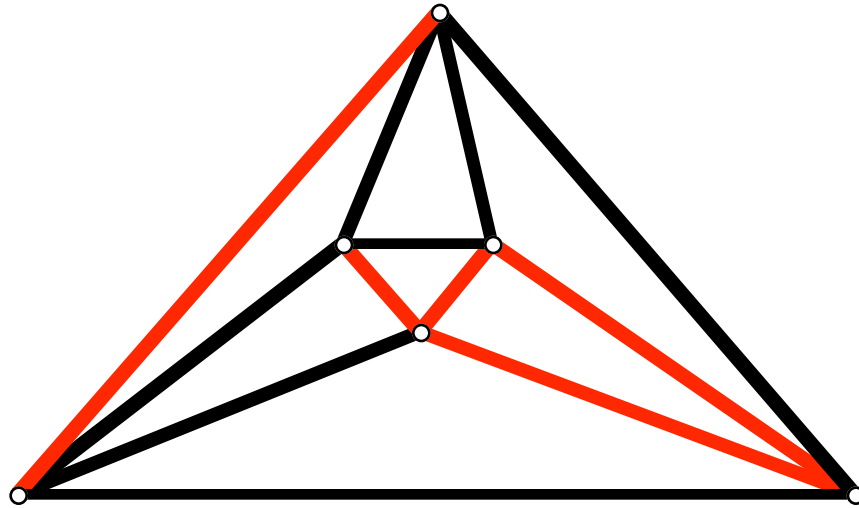
A wide variety of edge lengths can be used to construct a polyhedron with this graph (combinatorial type).

It seems to be of particular interest to have realizations of the graph with relatively few different edge lengths.

When edges of a plane 3-polytopal graph have been assigned different colors think of these as being different edge lengths for the edges of a 3-dimensional convex polyhedron realizing the plane graph in 3-space.

Note: Usually edge colorings require that two edges that meet at a vertex get assigned different colors. In our framework this requirement will not typically hold.

Plane triangulation, simplicial graph,
with its edges assigned (colored) one
of two colors, black or red.



Triangles are two kinds of combinatorially equilateral
triangles and two kinds of combinatorially isosceles
triangles.

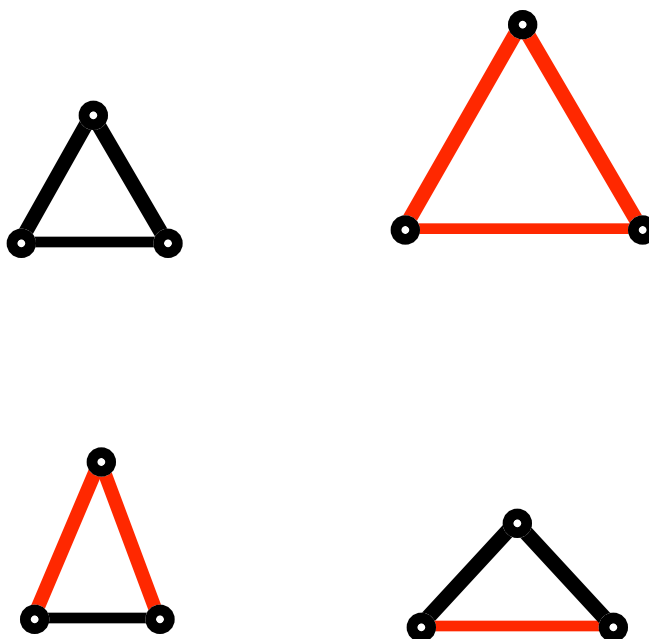
Four kinds of triangles with two edge lengths:

Black (b) equilateral, b^3

Red (r) equilateral, r^3

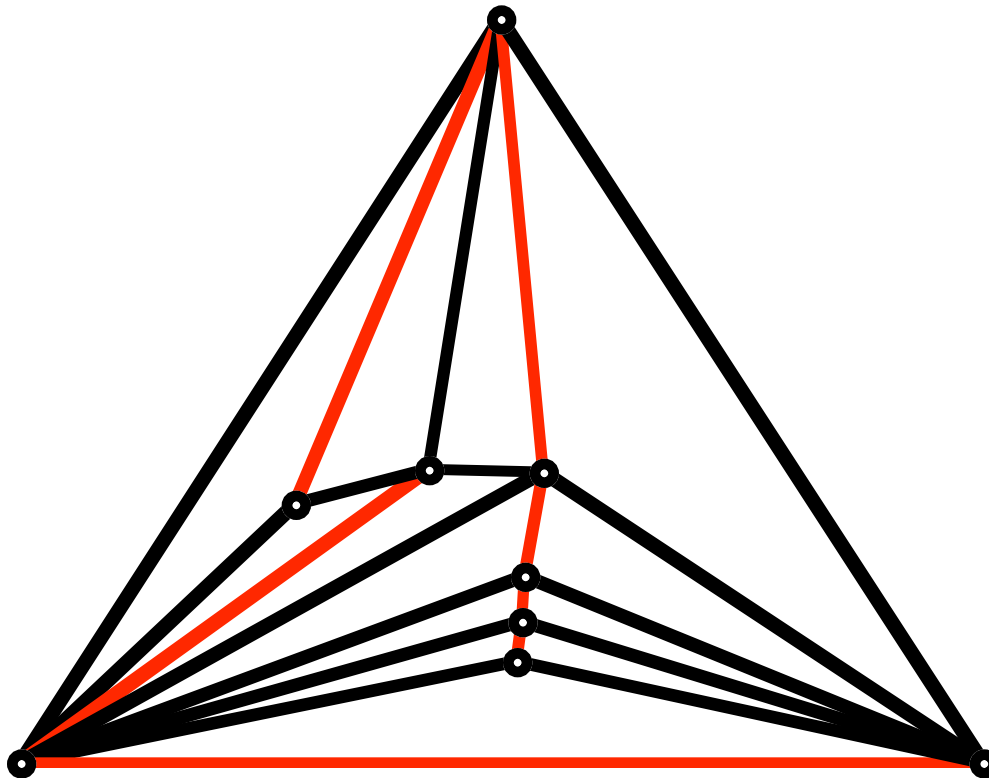
Black, black, red isosceles, b^2r

Red, red, black isosceles, br^2



What about realizing
polyhedra with only one
kind of isosceles triangle?

For example one might
have all b, b, r triangles.



Theorem:

Combinatorially, we can 2-color the edges of any triangulation so that every triangle gets colored with, say, two b edges and one r edge - all b, b, r triangles.

(Combinatorially "congruent" isosceles triangles.)

Proof idea:

Look at the dual graph, which is 3-valent, plane and 3-connected. Use Petersen's Theorem to find a perfect matching. (A set of disjoint edges including all the vertices.) The corresponding edges in the original graph would be the r -colored edges of the b, b, r isosceles triangles.

A geometrical version of this theorem fails.

This polyhedron can't be realized (geometrically) with congruent isosceles triangles!

Can't be realized with congruent isosceles triangles:

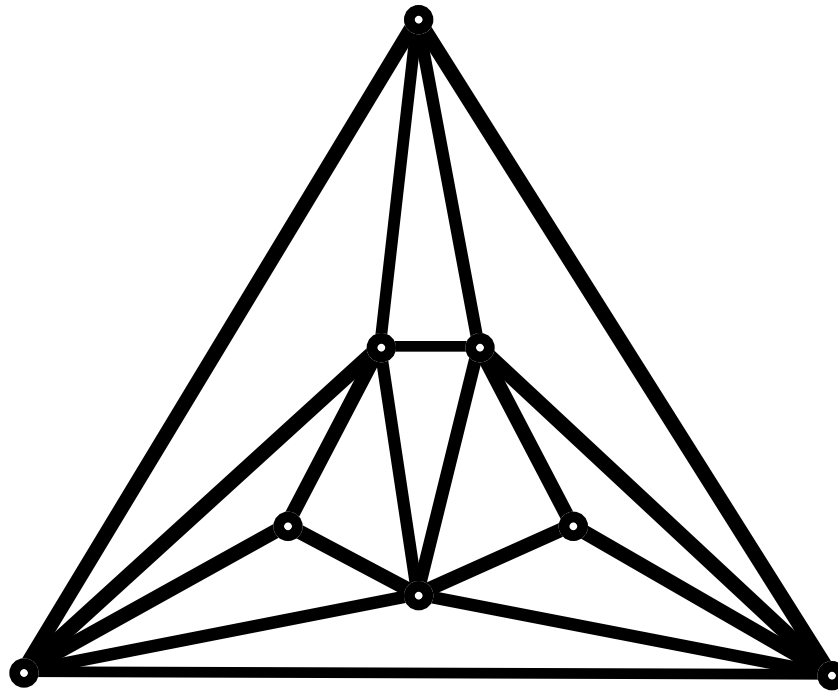


Figure (*)

What extra conditions
might guarantee
realizability?

(3 infinite families; are
there others?) Finite
number of sporadic
examples?

When a triangulation can be realized geometrically with congruent isosceles triangles of lengths b, b, r when can it also be realized with isosceles triangles of lengths r, r, b ?

Triangle inequality:

Note: Useful to assume $b + b > r$
and $r + r > b$

$$2r > b > r/2$$

$$2b > r > b/2$$

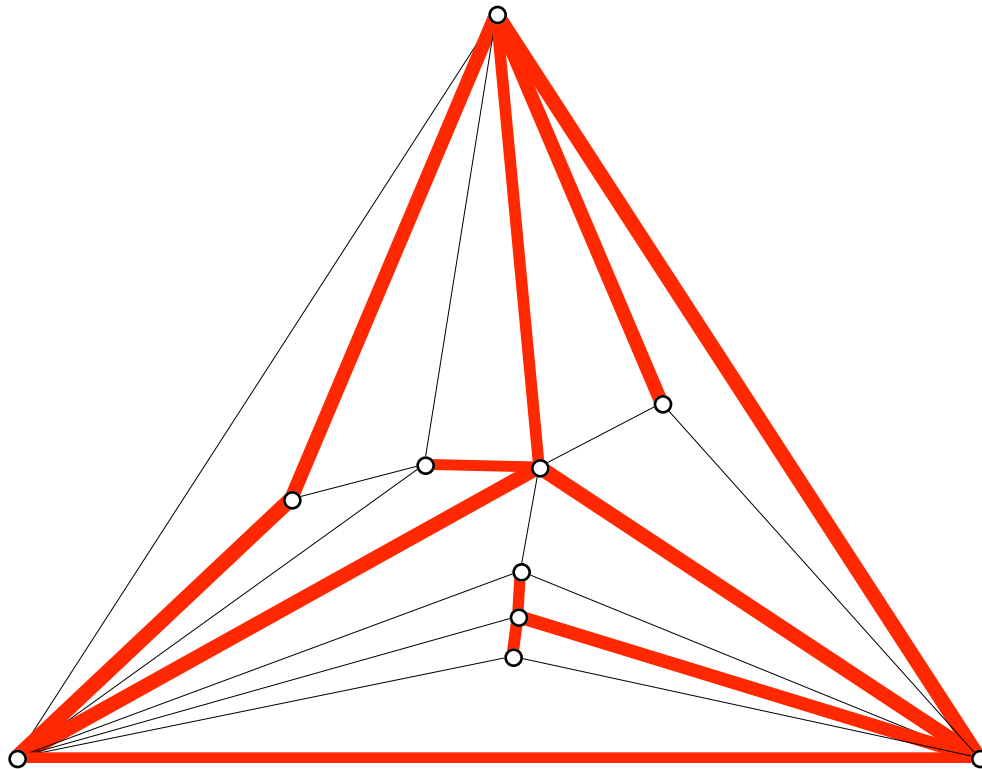
For models, integer values for b and r are often convenient!

In fact the combinatorial type (Figure (*)) above has several different combinatorially isosceles colorings but none can be realized geometrically.

Mate triangle coloring:

Equal numbers of r, r, b , and b, b, r triangles combinatorially.

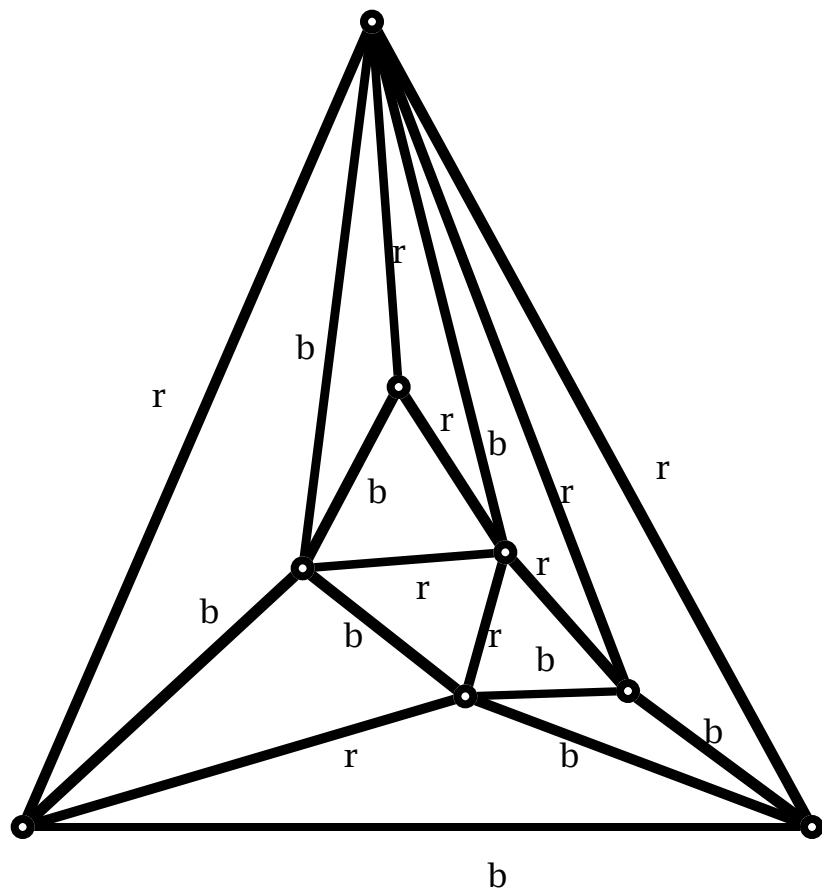
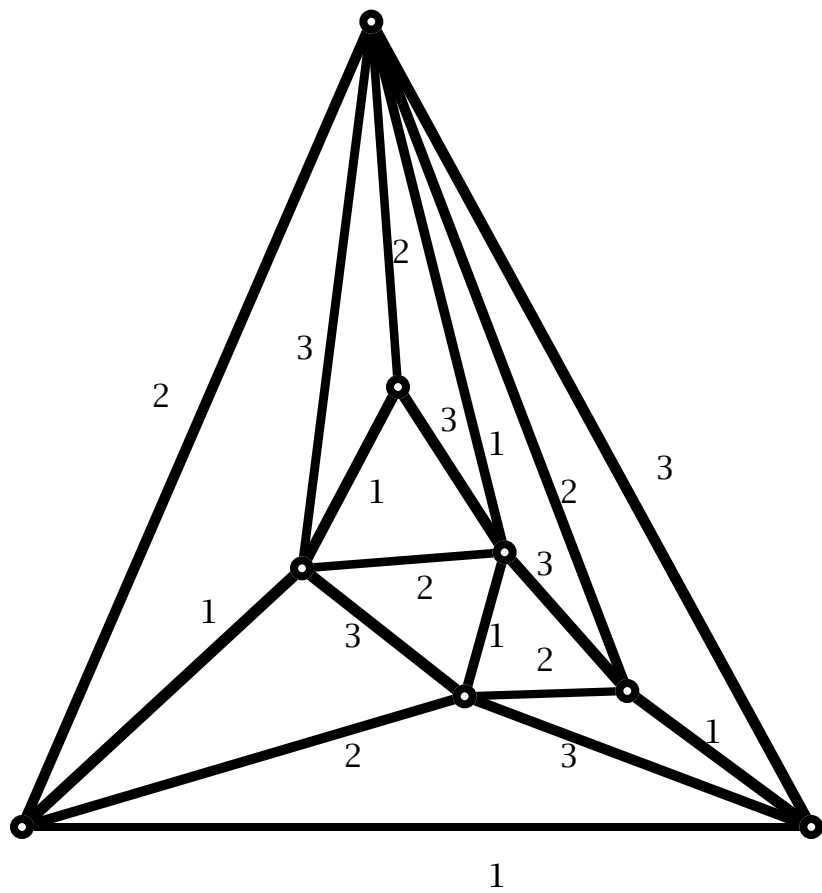
Such a coloring is possible (combinatorially) when the number of edges of the triangulation is even.



Open problem: Can any such coloring be realized by a polyhedron with the triangles of each type congruent to each other? (My guess is no, but I know of no counterexample.)

Proof idea:

By the 4-color theorem, a 3-valent 3-polytopal graph can be 3-edge colored with three colors, a so-called Tait coloring. Each of the 3 colors appears once at each vertex. Dualize this graph. If these colors are 1, 2, 3, change those colored 1 to blue, 2 to red, and half of the 3 edges colored 3 to black and half to red.



Note: If one omits the edges in the above triangulation colored 3, one gets a graph all of whose faces are 4-gons. This quadrilateralization may not be 3-connected as in the example above, but will be 3-connected when additional constraints are placed on the original triangulation.

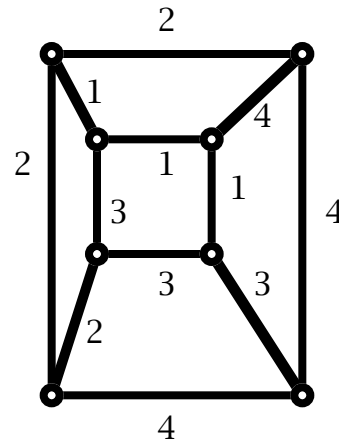
A Grünbaum coloring of a triangulation embedded in a surface is a coloring of the edges with colors 1, 2 and 3 so that each face (not necessarily 3-circuit) is colored with 1, 2, and 3.

Duals of 3-valent 3-polytopal graphs admit a Grünbaum coloring via the 4-color theorem and the associated 3-edge coloring (Tait coloring).

Useful tool for this circle of ideas (often rediscovered):

Kleinschmidt, P., 1978. Regular partitions of regular graphs. Canadian Mathematical Bulletin, 21(2), pp.177-181.

Plane 3-valent graphs can be "packed" with paths of length three; adjacent pairs of edges in the same face, but three edges are not part of the same face. Sometimes called a "Z" decomposition.



Middle edges of the paths form a perfect matching!

Generally, one can separate combinatorial and geometrical questions here.

The geometrical realization of the combinatorial questions is typically much harder.

Can the edges of any triangulation whose vertices are even-valent be colored with two colors so that every edge is in exactly one "equilateral" triangle?
(Ans. Seemingly, yes.)

Valences would obey:

$$v_4 = \sum_{s \geq 3} (s - 3)v_{2s}$$

s at least 4 (6-valent vertices have no restriction)

v_i is number of vertices of degree (valence) i .

One can also seek triangulations having every edge in at least one equilateral triangle where there are equal numbers of "equilateral" triangles of both kinds.

Barnette's Conjecture:
(Probably also known to
Tutte:)

Plane 3-valent, 3-connected
and bipartite graphs always
have a hamiltonian circuit.

Recent progress but still open. Note: even-valent plane
triangulations give rise to the family described above.

So the combinatorial results about congruent isosceles triangle triangulations, mate isosceles triangle triangulations, and triangulations where each lies in an equilateral triangle suggest exploring the geometrical realization aspects of these questions.

For what triangulation types are there convex 3-polytopes all of whose edges have at most two edge lengths?

Interesting special cases as previously noted!

More general
considerations:

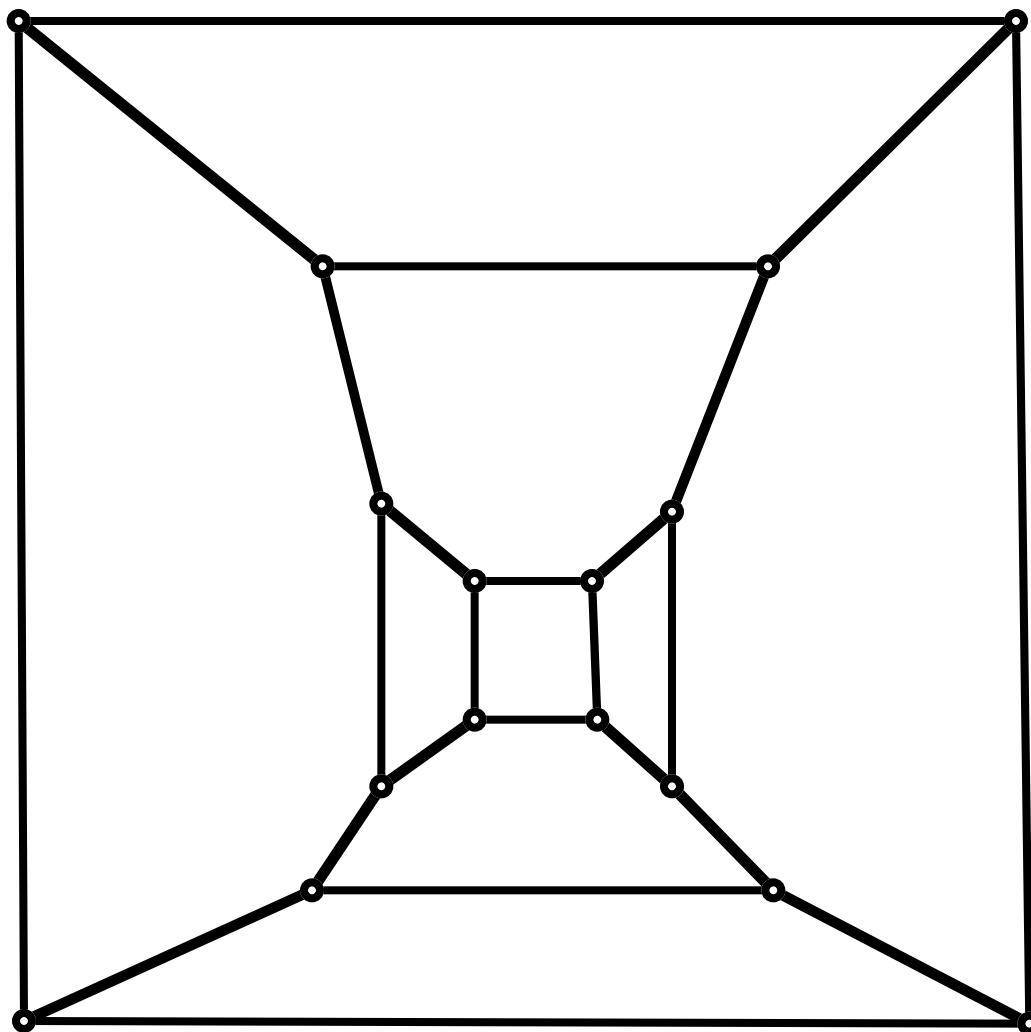
Problem:

Can every plane
triangulation be realized
by a convex polyhedron
with strictly isosceles
triangles for some
number of edge lengths?

Can every plane
triangulation be realized
by a convex polyhedron
with isosceles and
equilateral triangles for
some number of edge
lengths?

Interesting special case:

Duals of 3-valent graphs with six 4-gons and many hexagons may be an interesting family of triangulations to study!



Let $i(T)$ be the minimum number of edge lengths to realize triangulation T with strictly isosceles triangles.

Set $i(T) = \infty$ when such a realization is not possible.

Study the
behavior of $i(T)$.

Unfolding conjecture:

(Geoffrey Shephard) One can unfold any 3-dimensional convex polyhedron by cutting along the edges of a spanning tree and flattening the polygonal faces into a non-overlapping set of polygons in the plane (sometimes called a "net.")

Shephard's Conjecture is open even for all of 3-polytopes whose faces are triangles.

Special cases perhaps of interest:

3-polytopes whose faces are:

a. four 3-gons, and hexagons (h)

b. six 4-gons, and hexagons (h)

c. 12 5-gons, and hexagons (h)

(fullerenes - number of types grows rapidly with h - result of Thurston)

and duals of these solids which are triangulations with two types of vertices.

Thanks for your
attention!

Comments?

Questions?