Partitions, Compositions, Binary Strings and d-Cubes

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Binary strings of length k

length 1:

0
1

length 2:

00
01
10
11

length 3:

000
001
010
011
100
101
110
111

Counting in binary:

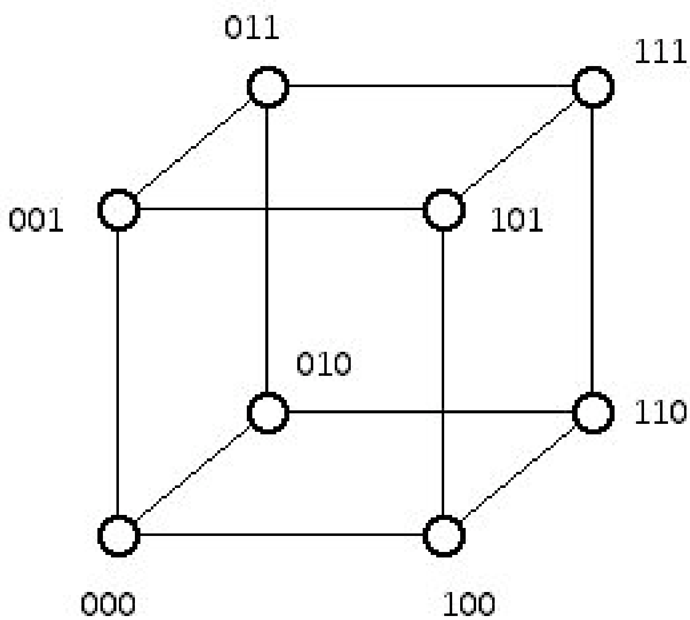
000
001
010
011
100
101
110
111

Note the large number of digit changes from one string to to the next.

The number of such changes is known as the Hamming distance

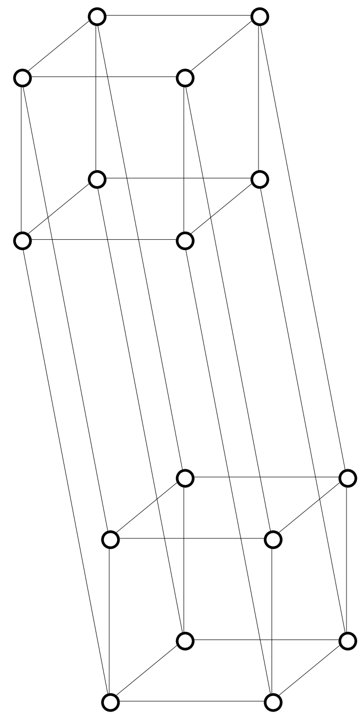
Gray code:

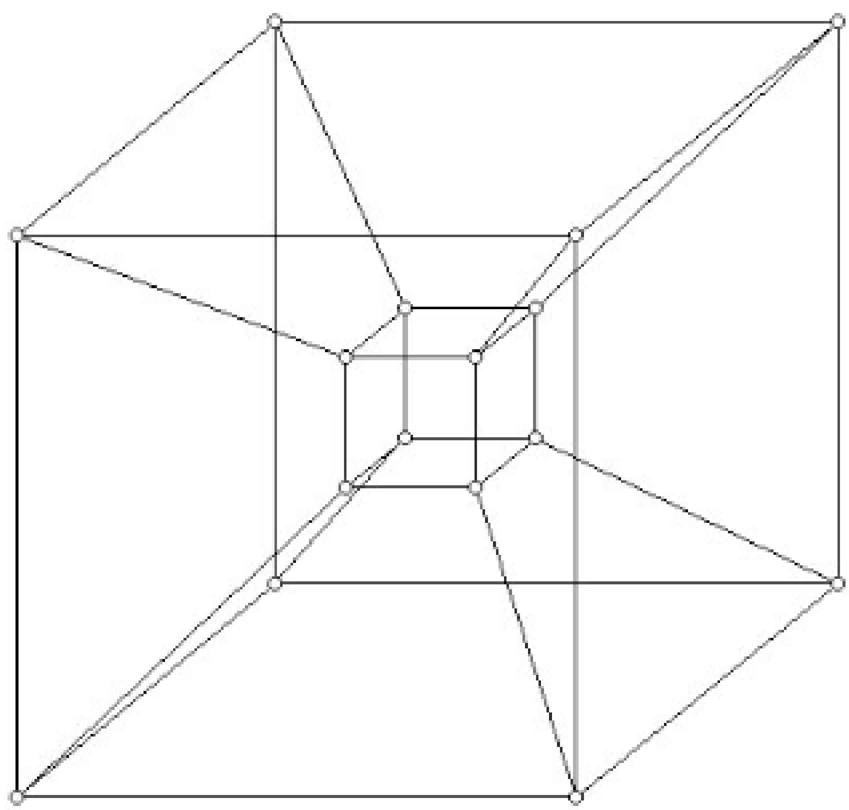
Cyclic sequence of binary digits that differ between consecutive pairs in only one place!

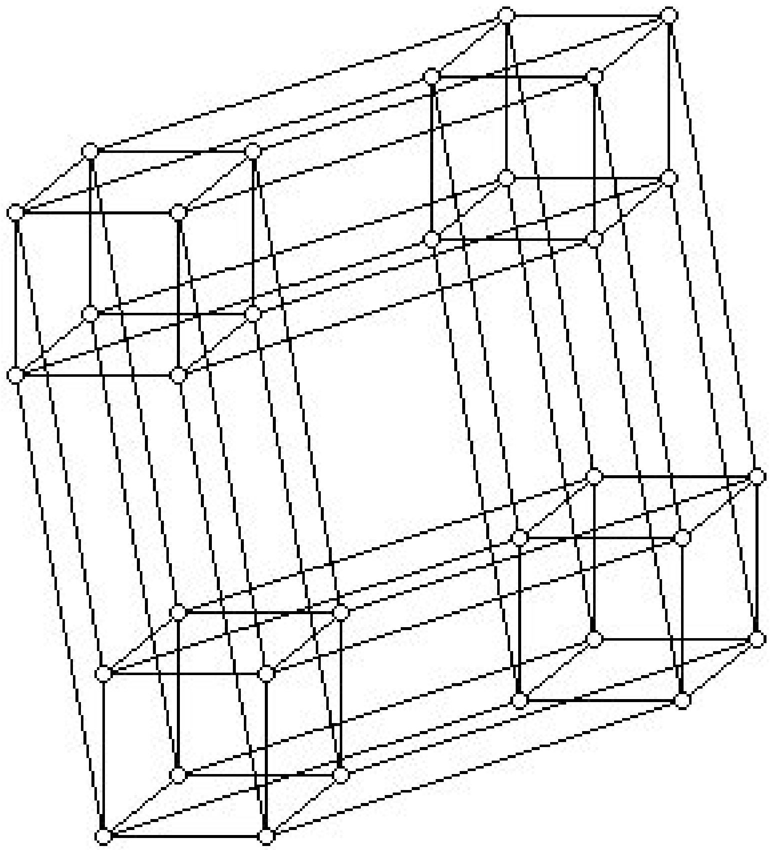


This gives rise to the d-dimensional cube! Gray code is any Hamiltonian Circuit on the cube.

One can count using binary digits or one can use binary strings as labels or names for things, without regard to their role as ways to count.







The partitions of a positive integer n are the ways of writing n as the sum of positive integers.

Example:

The partitions of 5 are:

5

4, 1

3, 2

3, 1, 1

2, 2, 1

2, 1, 1, 1

1, 1, 1, 1, 1

The numbers in a partition are called its *parts*.

Common interests are looking at even parts, odd parts, distinct parts.

Fact:

The number of partitions of n into distinct parts *equals* the number of partitions of n into odd parts.

Example:

5

4, 1

3, 2

3, 1, 1

2, 2, 1

2, 1, 1, 1

1, 1, 1, 1, 1

3 partitions above have distinct parts

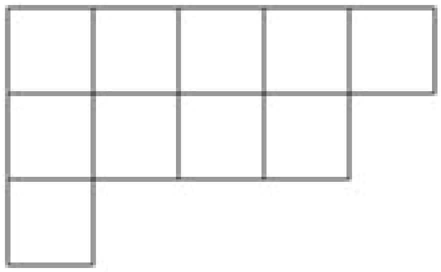
3 partitions above have odd parts

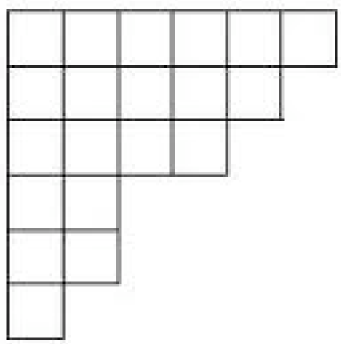
There are a variety of ways of representing partitions geometrically.

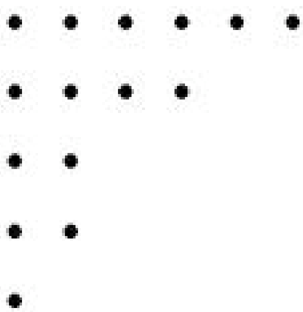
These are known as Ferrer's diagrams or Young's diagrams.

Norman Macleod Ferrers (11 August 1829 – 31 January 1903)

Alfred Young, FRS (16 April 1873 – 15 December 1940)

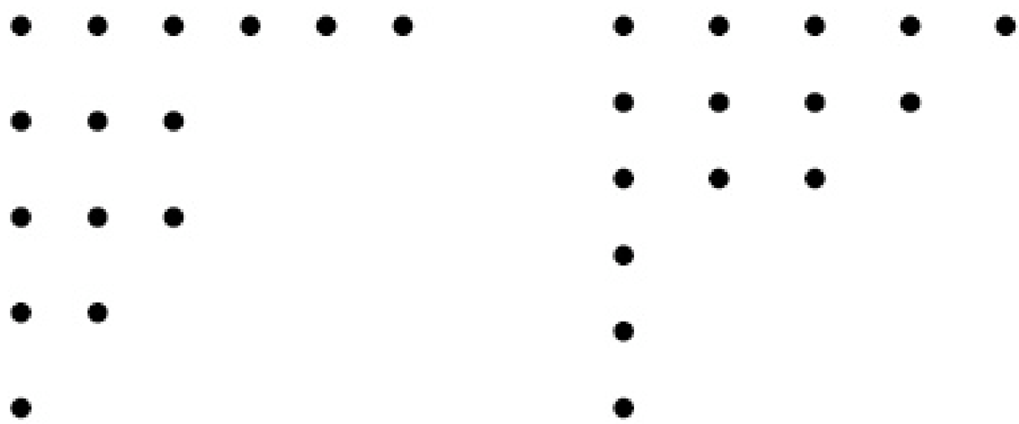




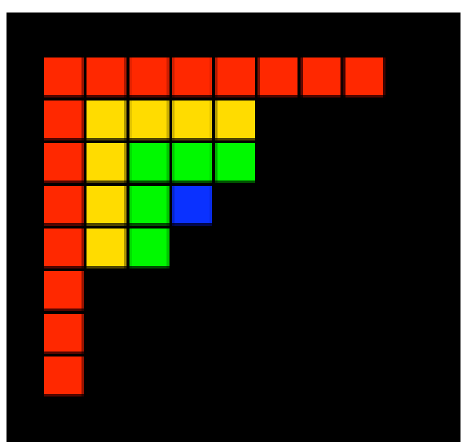


Given a Ferrer's diagram representing a partition one can construct from it another partition called the *conjugate* partition

Conjugate partitions:



Self-Conjugate partition:



Partitions of n which are self-conjugate equals the number of partitions of n into odd parts.

Example: $n = 12$

Self-conjugate:

6, 2, 1, 1,1, 1
5, 3, 2, 1, 1
4, 4, 2, 2

Odd parts:

9,3
7,5

11, 1

Compositions:

We take into account the order of the parts:

Compositions of 4:

4

3, 1

1, 3

2, 2

2, 1, 1

1, 2, 1

1, 1, 2

1, 1, 1, 1

8 compositions in all for n = 4

Theorem:

There are 2^{n-1} compositions of n

Percy Alexander MacMahon (1854-1929) helped pioneer "representations" for partitions and compositions.

MacMahon graph:

Example:

2 + 4 + 1 + 1 +5 = 13

(sometimes period used instead of a vertical bar)

--|----|-|-|-----

We code a composition of n with l parts as a binary string:

01 0001 1 1 0000

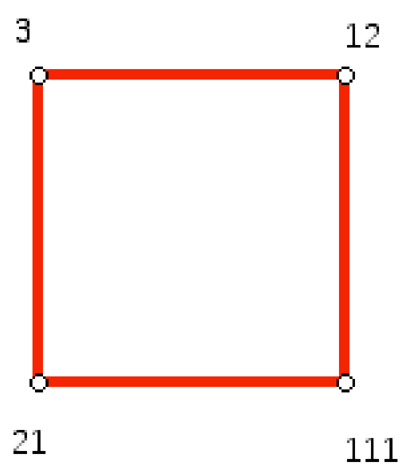
For ease in decoding spaces are used to separate parts.

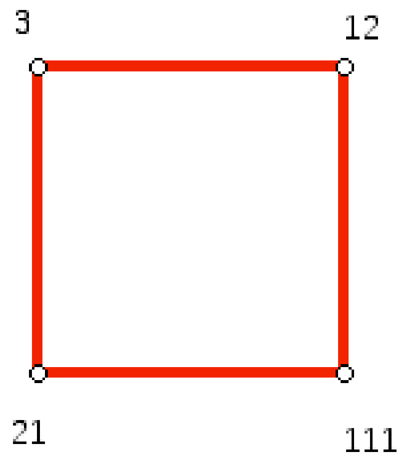
Note:

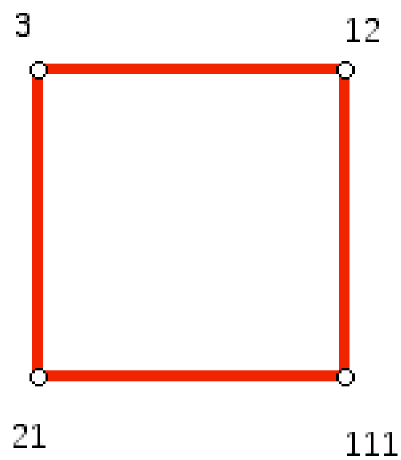
Given a string of 0's and 1's the ones represent plus signs in the composition.

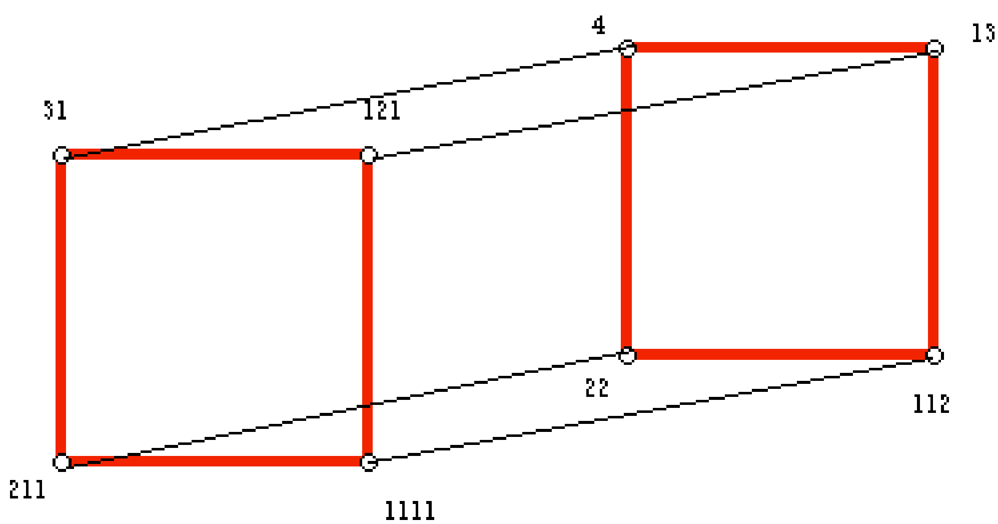
Hence, since we have (n-1) positions in which can be either 0 or 1 we have a proof that there are $2^{(n-1)}$ compositions!

Compositions and cubes:




3
12
21
111


3
12
21
111


4
13
31
121
22
112
211
1111

Application:

Classifying geometrical shapes

Code types of quadrilaterals by the
a. partitions of 4
b. compositions of 4

1+2+1 means the longest side is longer than than two equal sides which is longer than the 4th side.

Code the number of tetrahedra by

a. partitions of 6

b. compositions of 6

A {2, 2} quadrilateral:

More generally:

Given a geometrical problem with a parameter n, look at the number of inequivalent "objects" from a partitions and/or compositions point of view.

Thanks for you attention!

Good luck on your examinations!

Have a great summer!