Mathematics and CS Club

An Introduction to Mathematical Insights into Fairness

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Who are the scholarly experts on fairness?

Dating back to ancient times philosophers have discussed fairness issues and this tradition has been carried on into our times.

Aristotle wrote extensively on fairness.

He pointed out that one does not have to treat people exactly alike to be fair!

John Rawls (1921-2002)

Justice as Fairness

Rather surprisingly perhaps mathematicians have also systematically investigated many aspects of fairness.

One notable example is that Lewis Carroll the author of *Alice in Wonderland* but less famously being a professor of mathematics in England was an early contributor to the mathematical theory of elections.

Mathematics has looked at fairness issues related to:

a. Elections and voting

b. Fair legislative representation

c. Weighted voting

d. Cost sharing

e. Fair allocation (fair division - cake cutting)

f. School choice (market design)

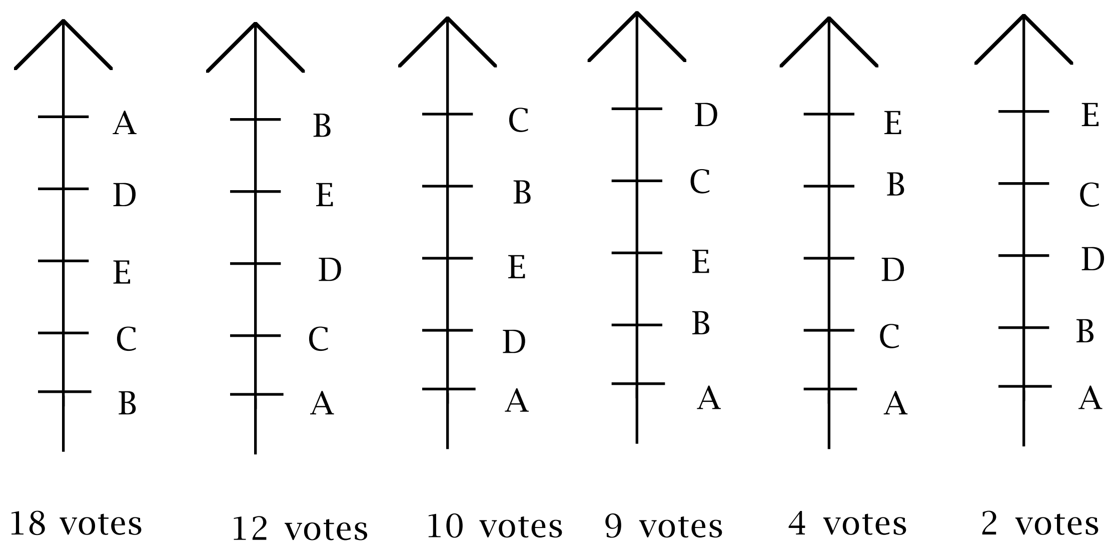
g. Fair vaccination programs

i. Gerrymandering

j. College admissions

Who has the best claim to win the following election?

Higher preferences towards the top:



Note that in most American elections one merely votes for one's favorite choice.

However using these ranked or ordinal ballots one can get more nuanced views about the choices/candidates from the voters.

One might, instead of asking for RANKED ballots, ask the voters to assign points to each of the candidates rather than rank them -cardinal ballots.

Numbers can be used to count and numbers can be used to measure.

ordinal numbers (natural numbers)

cardinal numbers (real numbers)

Some appealing methods to conduct elections:

1. Plurality (winner gets the largest number of first place votes)

2. Run-off (If no candidate has a majority, eliminate all but the top two vote getters and hold an election between them

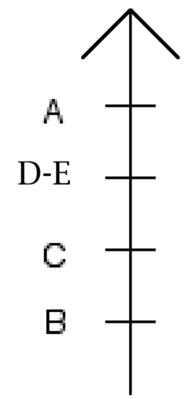
3. Sequential run-off (IRV - instant run-off voting) (If no candidate has a majority, eliminate the candidate with the lowest number of first place votes; transfer these votes to the other remaining candidate. Repeat until there is a single winner.

(To be used in NYC starting in 2021. Ballots will allow up to 5 choices.)

4. (Condorcet) Winner is the candidate, if there is one, who can beat all the other candidates in a two-way race.

5. (Borda Count) Given a ballet assign point to each candidate on the ballot in terms of how many candidates are below a given candidate on that ballot.

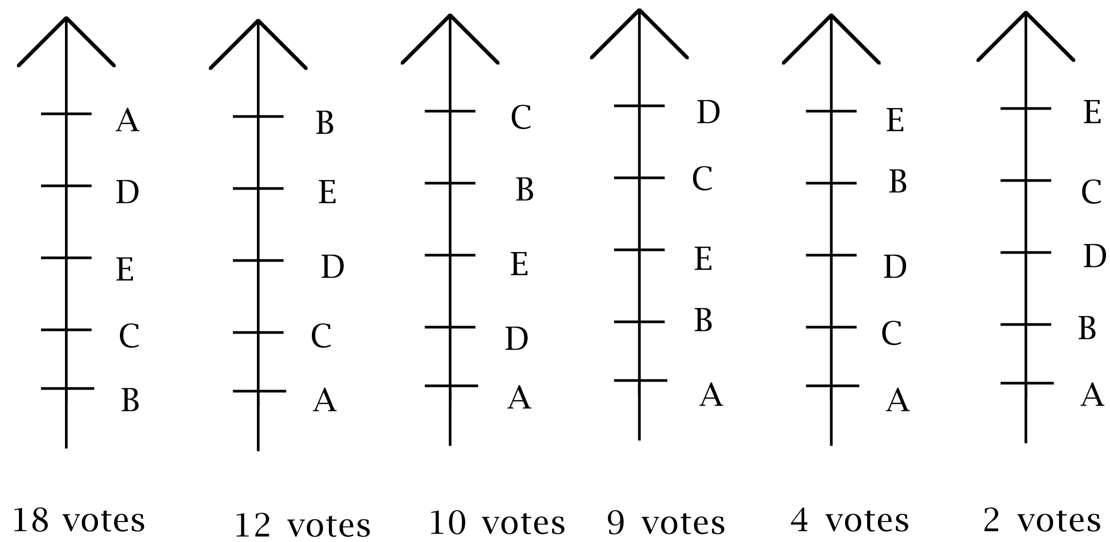
Example:



A gets 4 points
B gets 0 points
C gets 1 point
D gets two point
E gets two points

If 10 voters with this ballot, multiply by 10.

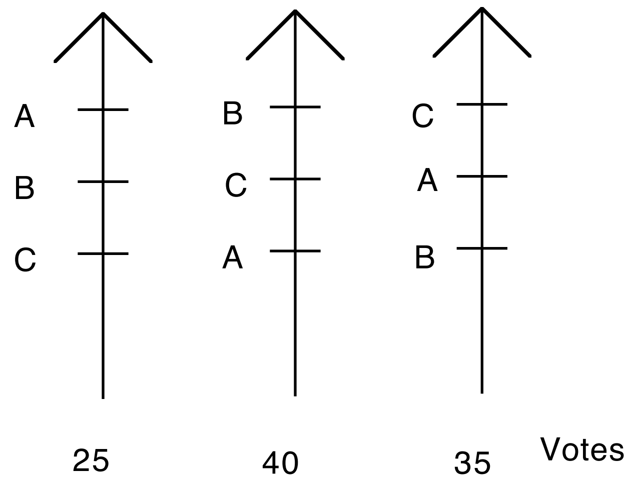
Another notation: A > D=E > C > B



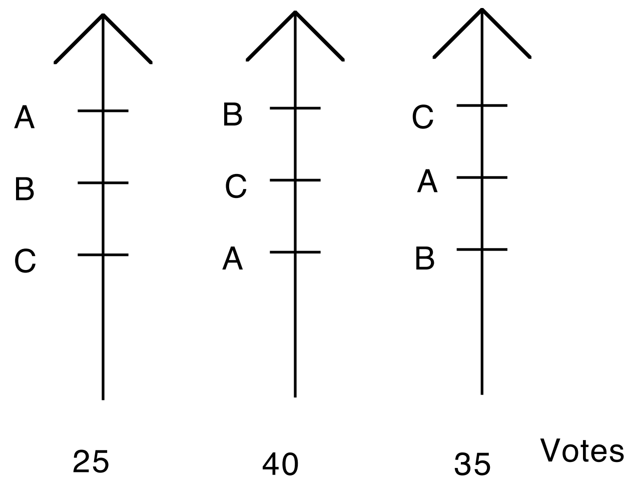
Perhaps surprisingly the 5 different methods just described give 5 different winners!

When a person wins an election perhaps it is less the "will of the people" rather than the method chosen to count the votes that matters!

Consequences of no Condorcet winner:



Note what happens if voting on items takes place sequentially in pairwise votes. Many real world legislatures work this way.



a. Vote on A vs. B; pit winner against C

A wins initially; C wins A vs C - C becomes law.

b. Vote on B vs. C; pit winner against A

B wins initially; A wins A vs B - A become law.

c. Vote on A vs. C; pit winner against B

C wins initially; B wins B vs. C - B becomes law

So how can one choose between different appealing methods?

Kenneth Arrow (City College graduate before CUNY existed) and winner of the Nobel Memorial Prize in Economics suggested the idea of seeing which nice FAIRNESS properties different methods obeyed and picking that method which obeyed the fairness rules one felt were important.

Examples of fairness rules:

1. Non-dictatorial

2. Non-imposed

3. Universal

4. Monotonic

5. Independence of irrelevant alternatives.

Arrow's Theorem:

There is no election method when one chooses among 3 or more candidates candidates using ranked ballots with ties allowed which obeys this list of fairness rules!

The mathematical model that Arrow builds involves:

a. Voters
b. Choices candidates
c. Ballot
d. Election decision method (a function mapping any election (individual choices) to a ranking (society choice).

More troubling and more general result.

When conducting elections with ordinal or point ballots (give each candidate some number of points from 0 to 99), does it ever help to misrepresent or "lie" about one's true feelings to help a particular candidate?

Voting of this kind is called *strategic*.

Satterthwaite-Gibbard Theorem:

When there are three or more candidates the only election decision method that cannot be manipulated is dictatorship!!

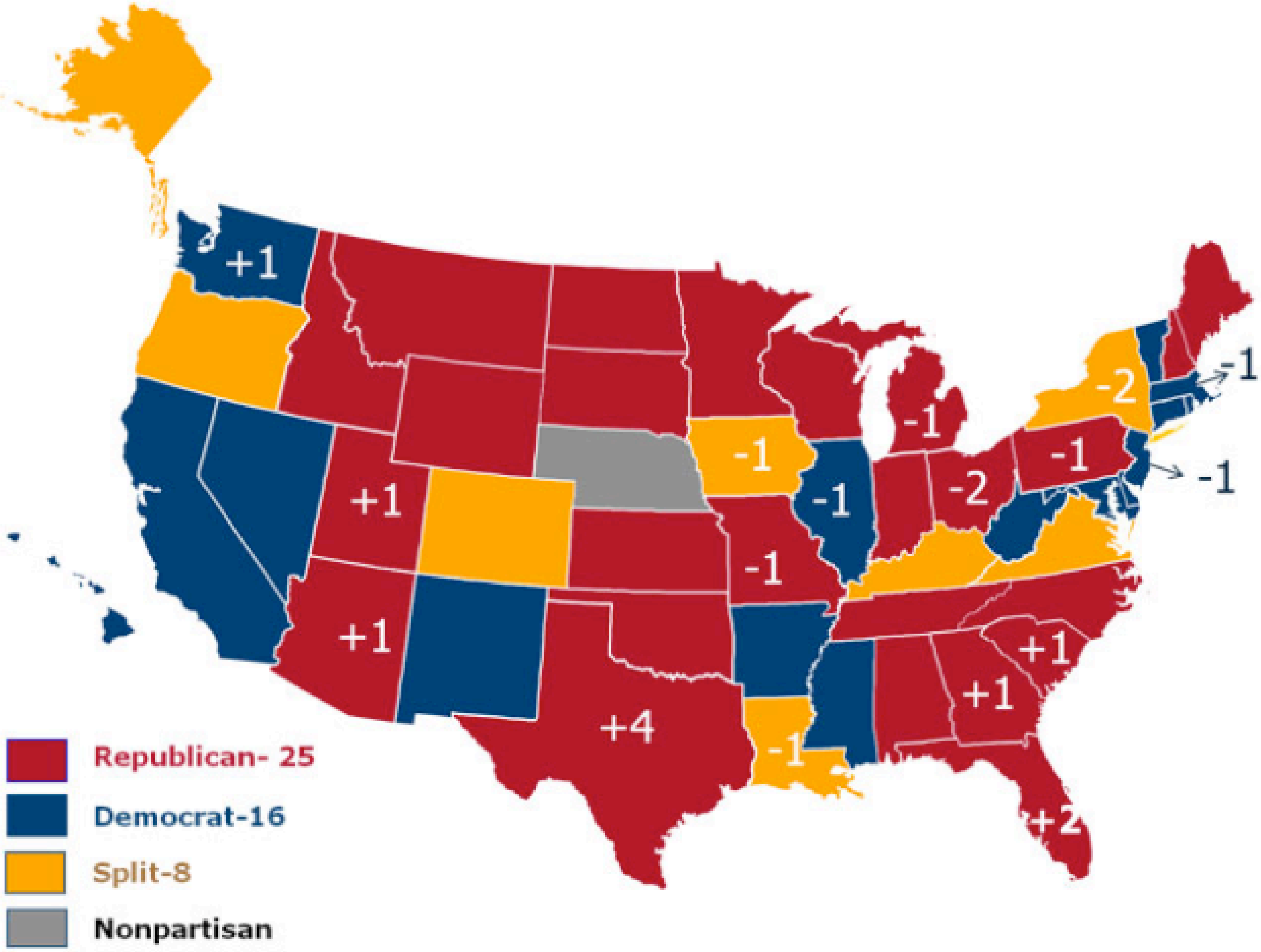
Apportionment:

Given parties or states with claims on the number of seats in a legislative body, say the 435 seats of the US House of Representatives, how can one determine the fair number of seats for the state or party?

The US Constitution requires that a Census be conducted every 10 years, and the Congress passed a law that using the Huntington-Hill apportionment method that seats be assigned to the 50 states, and the Census data is also used to determine the size of "block grants" to the States.

Due to attempts by the Trump administration for the census not to count those who are not citizens (contrary to what was done in the past) the reapportionment is currently delayed because the Census was not finished on time. Worse, many people were intimidated not to participate for fear of their immigration status.

Changes in 2010:


+1
-1
-2
-1
-1
-1
-1
-2
-1
+1
-1
+1
+1
+1
+4
-1
+2
Republican- 25
Democrat-16
Split-8
Nonpartisan

The current method used to apportion the US House of Representatives is based on computing, giving each state one seat and then additional seats one at a time using a table derived from the state populations involving using the geometric mean.

Geometric mean of a and b is $\sqrt{(ab)}$

Question: County Z has 5 towns with populations of 900,000, 500,000, 500,000, 400000, and 200,000. What might be a good set of weights and a quota for a county legislature with 5 players?

Have each county's representative case a number of points, 1 point for each 100,000 people!

Weighted voting game:

[13; 9, 5, 5, 4, 2]

Players names are 1, 2, 3, 4, and 5

Minimal winning coalitions:

{1, 2} {1, 3} {1, 4} {2, 3, 4}

Minimal winning coalitions:

{1, 2} {1, 3} {1, 4} {2, 3, 4}

Player 5 has NO power! Player 5 is never a member of any MINIMAL winning coalition. That is, a group of players who if any player is deleted from that coalition can no longer take action (win). Players with positive weight who are never a member of a minimal winning coalition are called *dummies* in the weighted voting literature.

The reason why Nassau and Suffolk County now have county legislatures rather than using the weighted voting they used to use is that the ACTUAL weights gave rise to dummy players! John Banzhaf won the court case making this practice unconstitutional, and now for upstate counties weights must be proportional to Banzhaf Power!!

Example:

[5; 4, 3, 2] Three players named 1, 2, and 3 who cast 4, 3, and 2 votes respectively. The 5 is called the quota. Players with combined weight of 5 are needed to take action.

Is Player 1 twice as powerful as Player 3 because 4 is twice 2?

[5; 4, 3, 2]

Which coalitions (collections) of players can take action?

Minimal winning coalitions - no subset of a minimal winning (MW) coalition wins:

{1,2}, {1,3}, {2,3}

Given [5; 4, 3, 2], we have total symmetry here for the MW members. The MW coalitions are:

{1,2}, {1,3}, {2,3}

so it should be apparent that in this game all three players have equal influence!!!

An isomorphic game would be:

[2; 1, 1, 1]

because its minimal winning coalitions are also:

$$\{1,2\}, \{1,3\}, \{2,3\}$$

Power indices: (Variants differ in using all winning versus MW coalitions) (Names are not standardized.)

a. Coleman

b. Banzhaf

c. Shapley-Shubik

d. Deegan-Packel-Johnston

[5; 4, 3, 2]

MW: {1,2}, {1,3}, {2,3}

Coleman:

1 is in two coalitions
2 is in two coalitions
3 is in two coalitions

So 1 has 2/6 as a power; 2 has 2/6 as a power; 3 has 2/6 as a power!

Look at the pattern of Yes and No votes of the 3 players:

YYY wins
YYN wins
YNY wins
YNN loses
NYY wins
NYN loses
NNY loses
NNN loses

Underlines show when a Yes changed to a No changes a win to a loss. So of the underlined items each player has 2 out of a total of 6. (This is Banzhaf Power.)

So each player has equal Banzhaf power.

Note: We only look for "pivots/swing," that is changes when a sequence of Y's and N's wins, and changing a Y to an N makes a win a loss.

It turns out that looking at situations where a pattern yields a loss and changing a No to a Yes wins, just doubles the number of pivots/swings because we are computing a ratio.

Template for computing the Banzhaf power for a game with 4 players:

This pattern holds for n at least 2. n players $2^{\mathbf{n}}$ lines in the table.

Pattern: Column 1, 8 Y's 8 N's; Column 2, 4 Y's, 4 N's, 4 Y's, 4 N's'; Column 3, Alternate 2 Y's, 2 N's, etc.

Y	Y	Y	Y
Y	Y	Y	N
Y	Y	N	Y
Y	Y	N	N
Y	N	Y	Y
Y	N	Y	N
Y	N	N	Y
Y	N	N	N
N	Y	Y	Y
N	Y	Y	N
N	Y	N	Y
N	Y	N	N
N	N	Y	Y
N	N	Y	N
N	N	N	Y
N	N	N	N

Shapley-Shubik Power Index

Consider all orders of the players voting. Reading from left to right give a pivot/swing point to the first player whose vote the sum of weights over the quota.

[5; 4, 3, 2]

1 *2* 3
1 *3* 2
2 *1* 3
2 *3* 1
3 *1* 2
3 *2* 1

Pivot player is shown in italics - second in every case for this example.

Hence:

Player 1 has 2 pivots out of 6; power 1/3

Player 2 has 2 pivots out of 6; power 1/3

Player 3 has 2 pivots out of 6; power 1/3

Remember that 2/6 is the same fraction as 1/3.

Banzhaf was not trained as a mathematician. He was trained in the law. He is most famous for winning cases against tobacco companies that smoking is harmful to one's health.

He also won a Supreme Court decision which overturned the use of weighted voting in Nassau County because there were players with NO Power!

Nassau and Suffolk now have legislatures rather than weighted voting but most upstate NY Counties have weighted voting procedures.

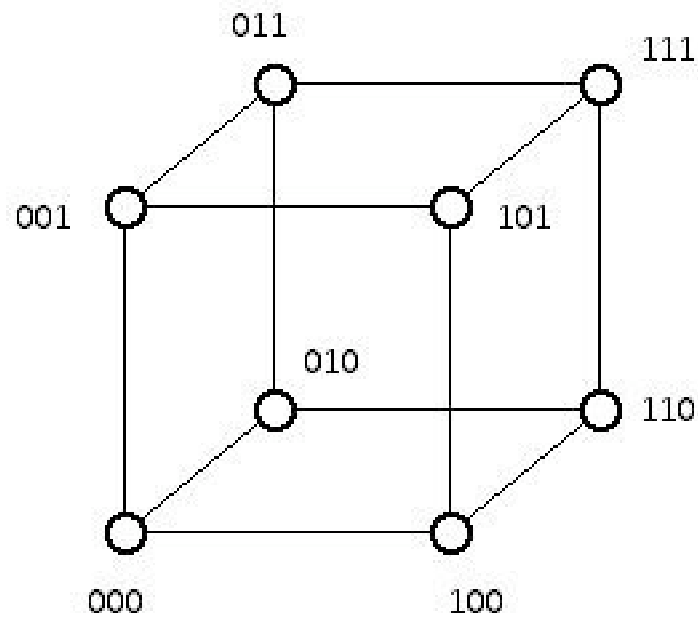
In NYS weights must be assigned to the players in the weighted voting games for county governments so that the Banzhaf Power is proportional to the population of the players involved.

Pattern of Yes/No for lines in a Banzhaf power table for 3-players "corresponds" to the labels needed for a 3-dimensional cube:

NNY,YNY,NYY,YYY(top)

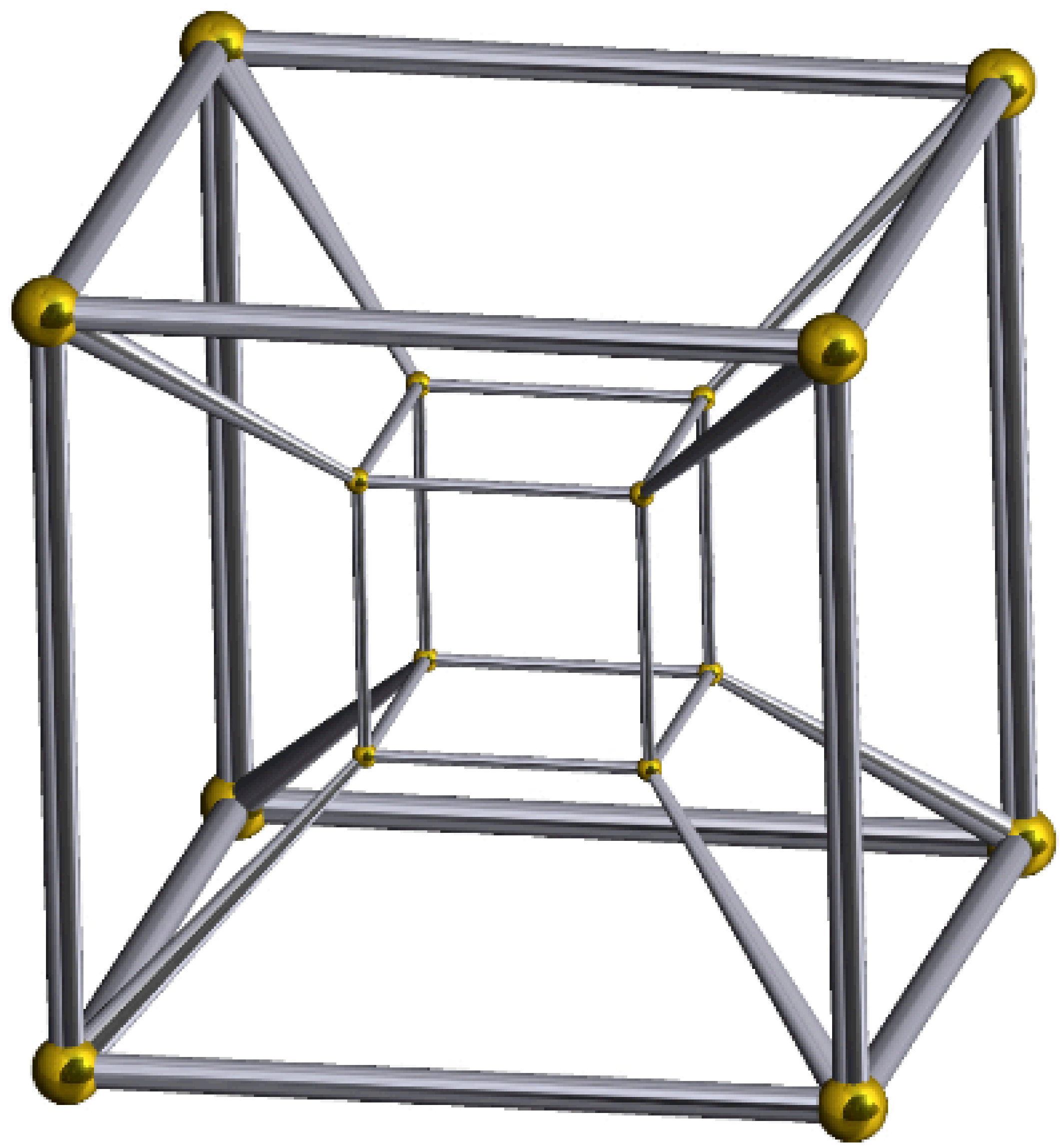
NNN,YNN,NYN,YYN(bottom)

Think of N as a 0 and Y as a 1:



Three-cube made from two 2-cubes! Top layer all entries end in 1; bottom layer all entries end in 0!

Banzhaf table for 4 players correspond is obtained by pasting together two copies of a 3-cube to get a combinatorial 4-cube.



Thanks for listening!

Questions? Comments?

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