Representing Convex Three-Dimensional Polyhedra in the Plane

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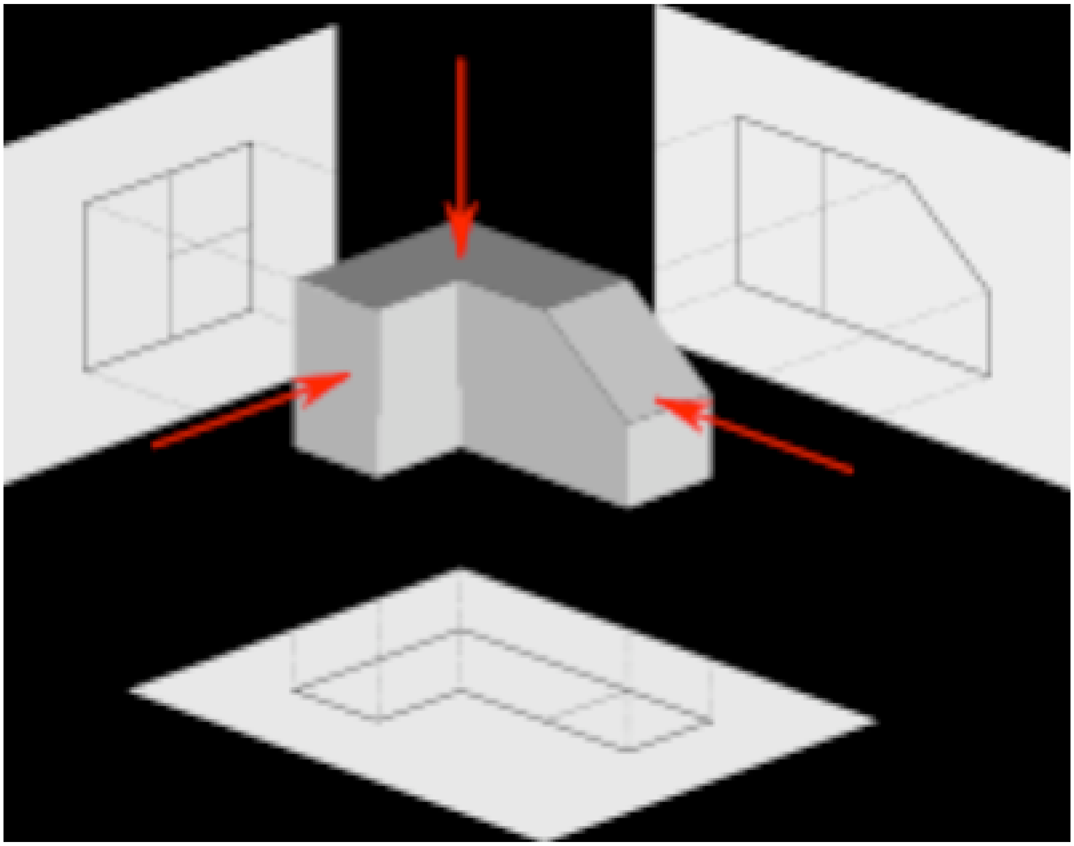
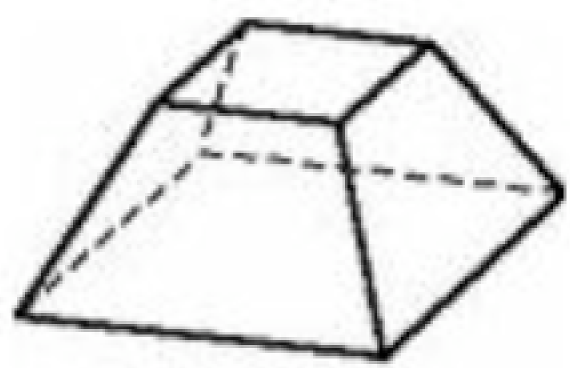
Representing polyhedra:

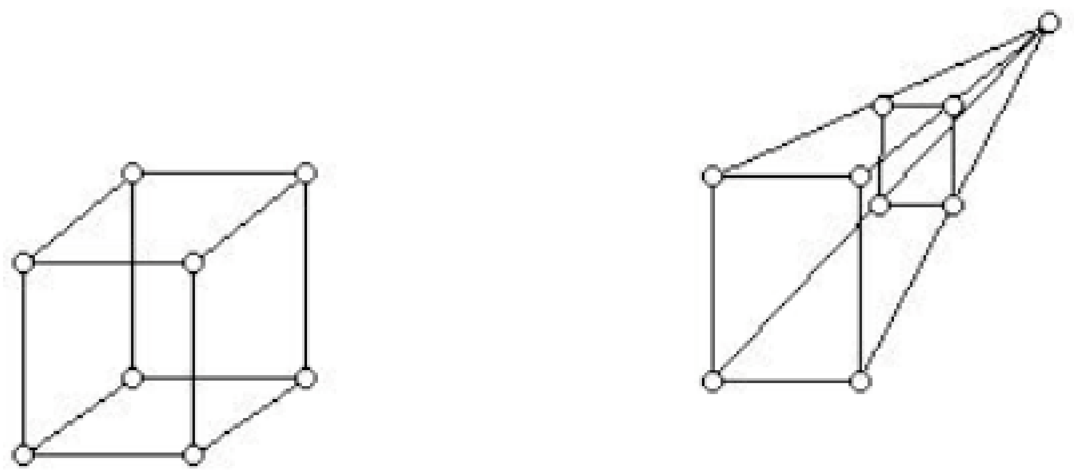


(Courtesy of Wiki)

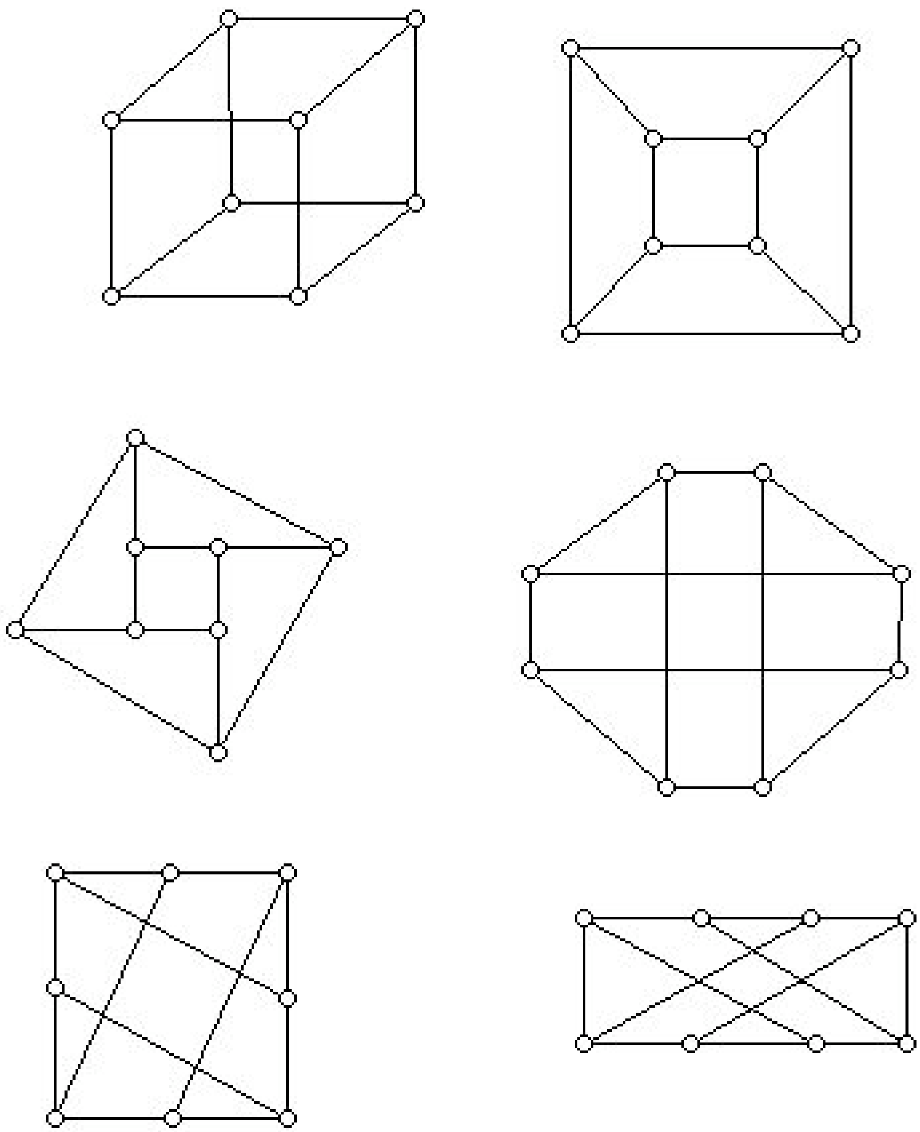
Hidden line drawing in the plane of a truncated square pyramid:



Isometric drawing of a cube and a perspective drawing of a cube:



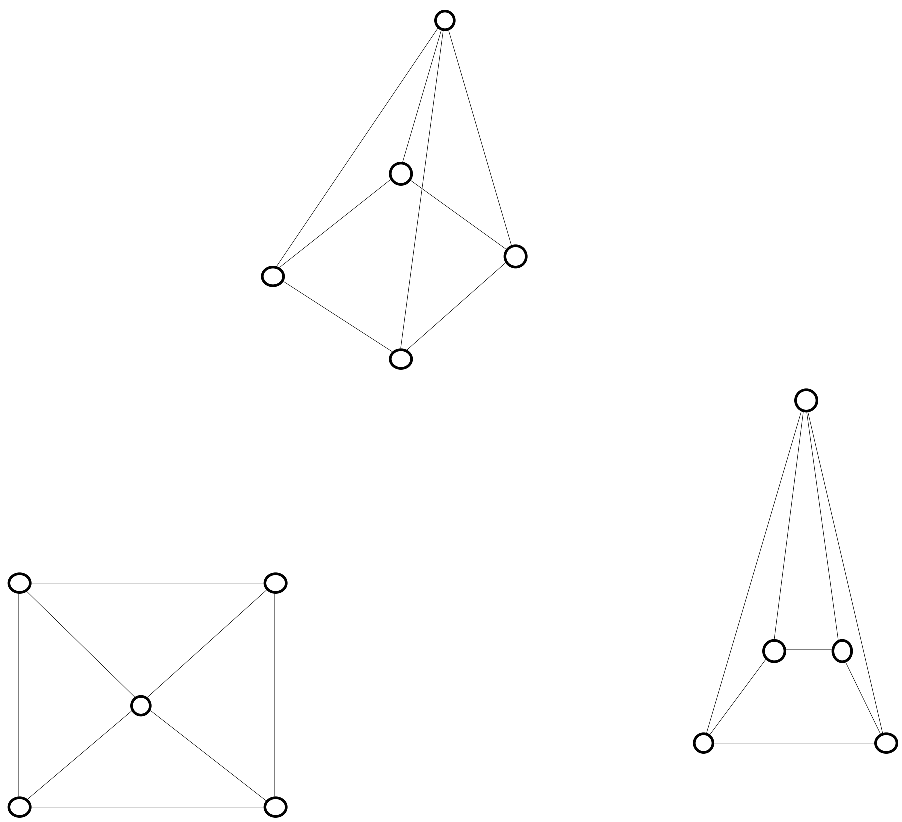
Graphs of the 3-cube:



Theorem: (Cauchy)

The graph of every convex 3-dimensional polyhedron is planar.

Any face can be the infinite face in a plane drawing:

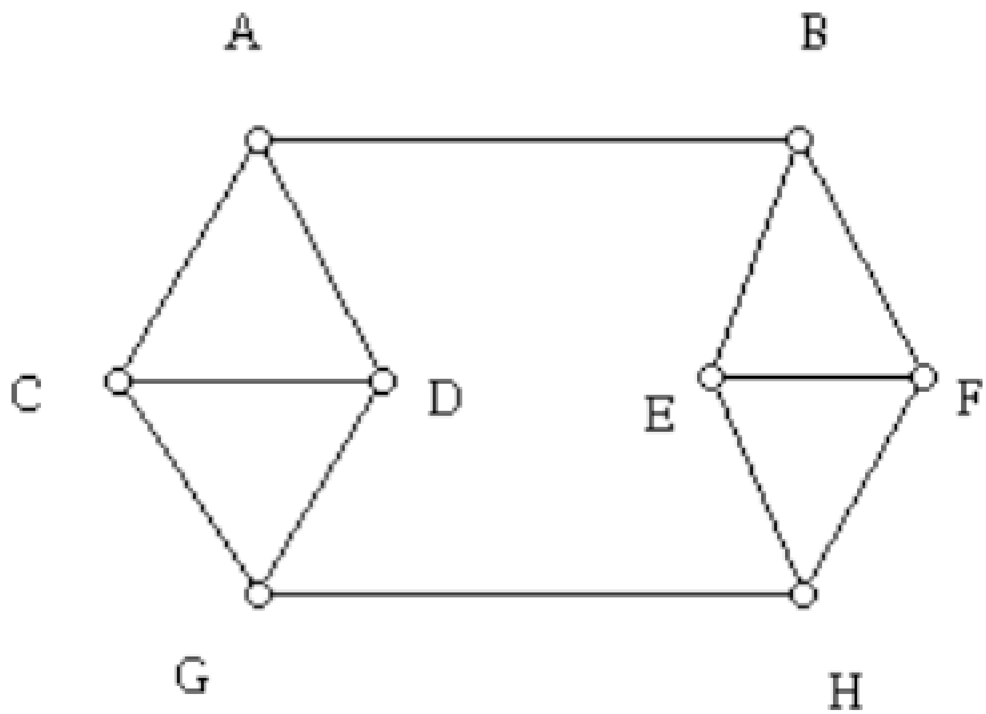


One can think about different metrical realizations of the same graph

Example:

One can have a "3-cube" consisting of six congruent squares (the regular hexahedron) or the "3-cube" which arise from truncating a square pyramid whose faces are equilateral triangles.

Is there a 3-dimensional polyhedron with this diagram as its graph?



Ans:

Non-convex: Yes!

Convex: No!

Steinitz-Grünbaum-Motzkin:

A graph G is the vertex-edge graph of a 3-dimensional convex polyhedron if and only if G is planar and 3-connected.

Significance:

Combinatorial properties of 3-polytopes can be investigated in 2-dimensions using a special class of plane graphs.

Key tools:

a. Euler's Polyhedral Formula

$V + F - E = 2$

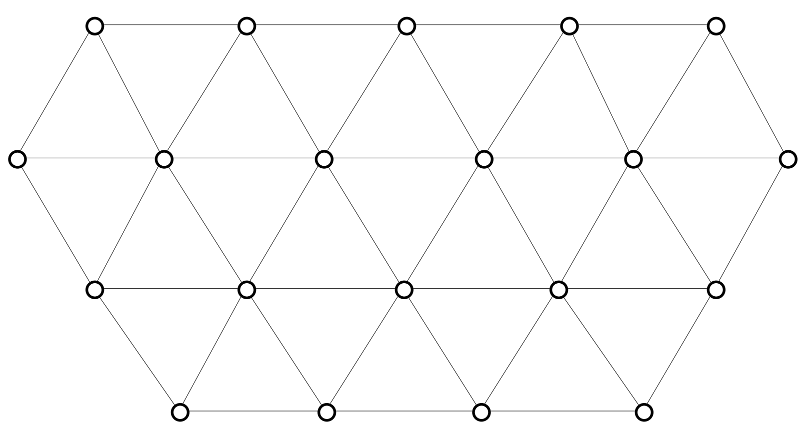
(which can be proven using Descartes Theorem or using graph theory methods)

Descartes Theorem:

(The deficiency at a vertex v of a convex polyhedron is 360º - sum of the face angles at v.)

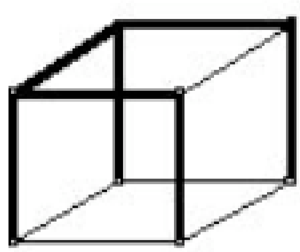
The sum of the deficiencies of the vertices of a convex 3-polytope is 720 degrees.

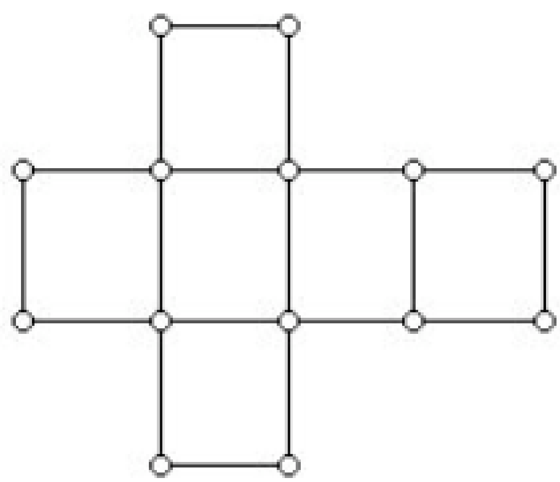
Can this polytope be realized metrically with many "nice" triangles?



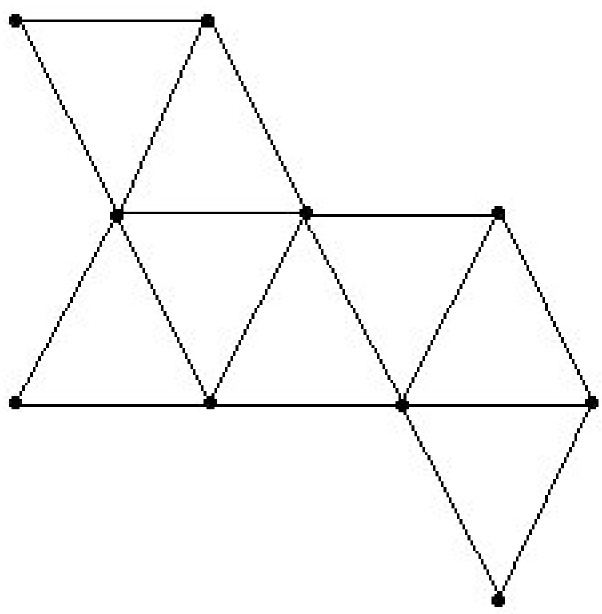
Dürer:

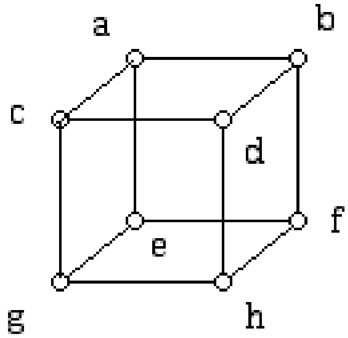
Albrecht Dürer seems to have been first to draw diagrams arising from cutting edges of polyhedra to represent them in the plane:





A "net" which can be folded to a convex or non-convex polyhedron:

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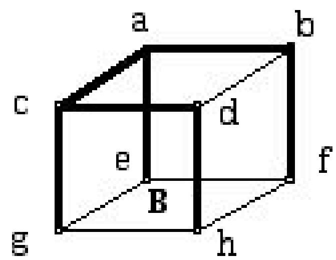
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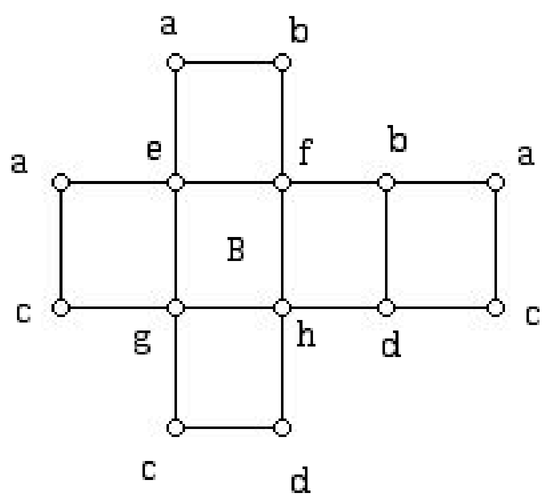
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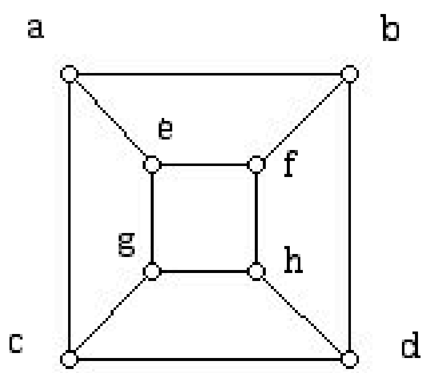
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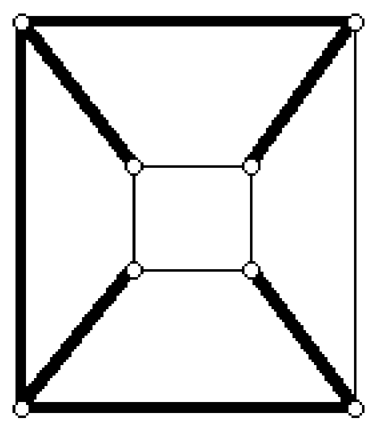
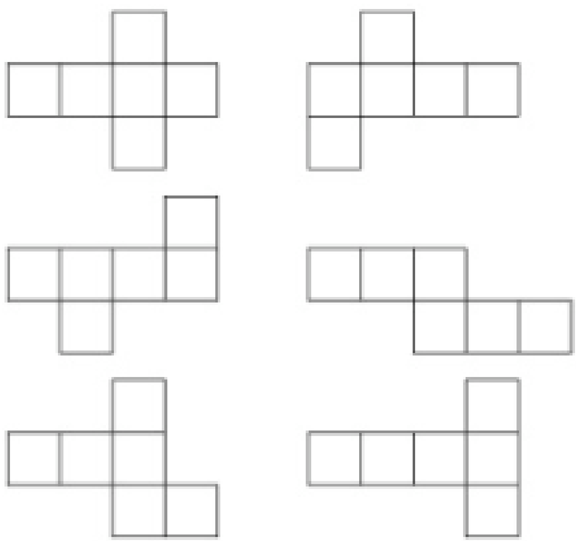


Eleven nets of the cube:



Shephard's Conjecture:

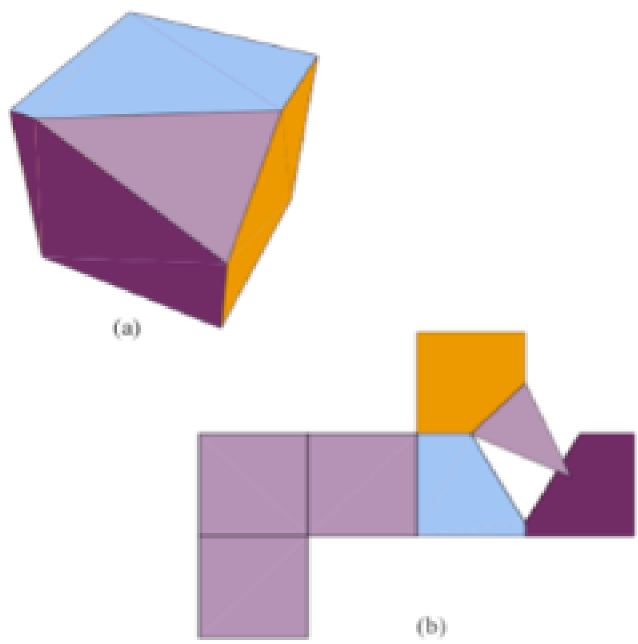
(Shephard, G. C. "Convex Polytopes with Convex Nets." Math. Proc. Camb. Phil. Soc. 78, 389-403, 1975.)

It is possible to cut along edges (of a spanning tree of the graph) of 3-polytope and unfold the polytope to a plane simple polygon.

The result is often called a non-overlapping "net."

Overlapping net:

Cube with one corner truncated



(Courtesy of Joe O'Rourke; Wolfram)

Alexandrov's Theorem (1948):

Every simple plane polygon can be "folded" to a unique convex polyhedron (or double covered convex polygon).

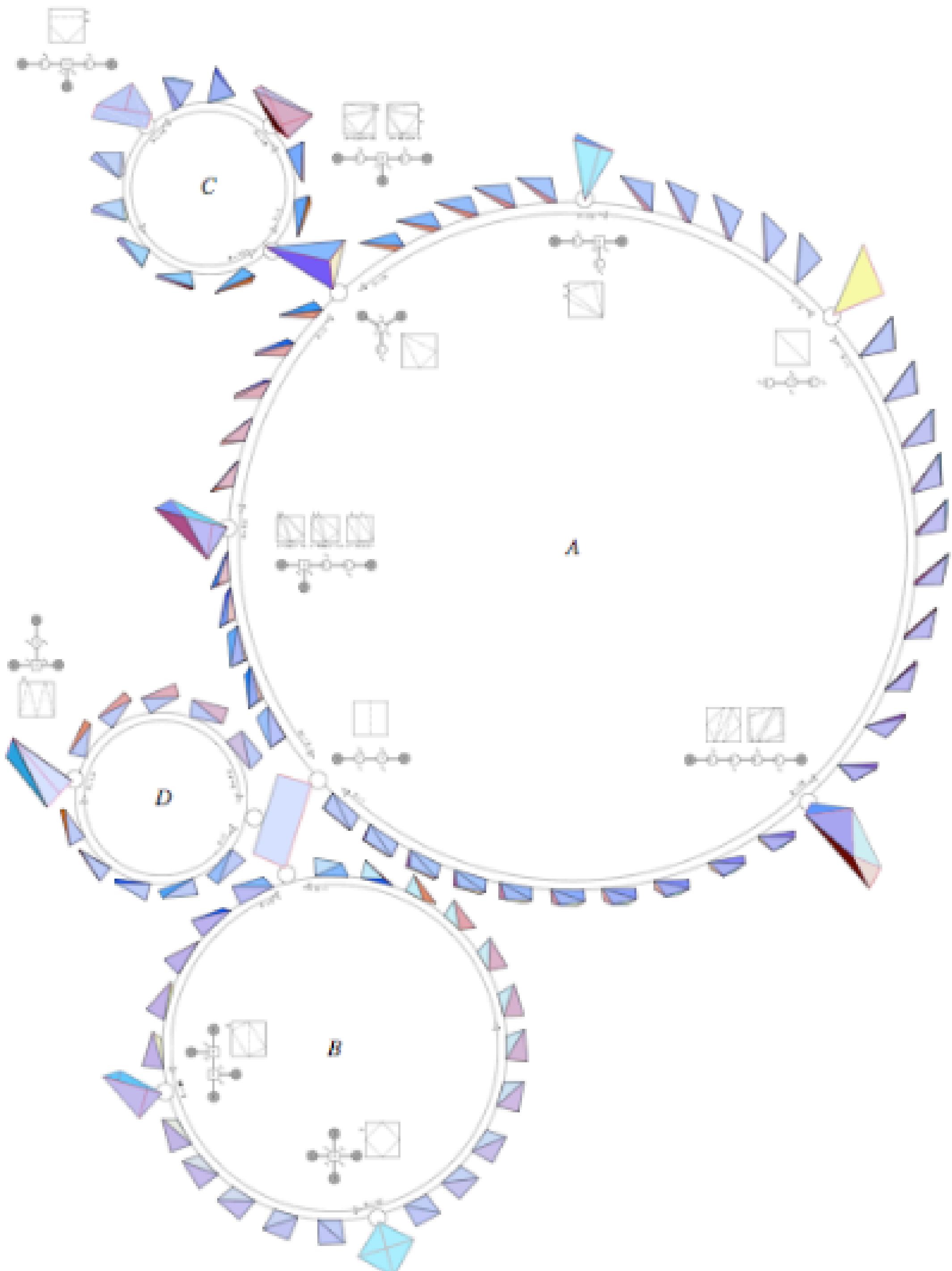
What is meant by "folded?"

Alexandrov "gluing":

a. The entire perimeter of the polygon is "used up." Many points can be identified to the same point in the gluing and a single point can be matched with itself.

b. The gluing creates angles with no more than 360º. (This is a condition on the angle sum at the points pasted together.)

c. The gluing results in a topological sphere or a double covered polygon



(Alexander, Dyson, and O'Rourke; 2002)

The following are foldable from a square:

* tetrahedra

* two types of pentahedra

* hexahedra

* octahedra

** triangle
** square
** rectangle
** pentagon

References:

E. Demaine and J. O'Rourke, Geometric Folding Algorithms: Linkages, Origami, and Polyhedra, Cambridge U. Press, NY, 2007.

J. O'Rourke, How to Fold It, Cambridge U. Press, NY 2011.