Graphs from Rectangles

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At its core mathematics is concerned with the study of patterns of different kinds. What is perhaps surprising is that changing points of view often show that patterns involving things we regularly see around us are not always as fully explored as we might think, especially in situations where the patterns emerge from looking at and exploring the properties of geometrical objects.

The basic objects that we will consider are rectangles drawn in the plane in a special way (Figure 1). Notice that the sides of the rectangles are either horizontal or vertical. Such rectangles are often said to be *axis-aligned*, since their sides are parallel to either the x- or y-axis when drawn in an (x, y) coordinate system.

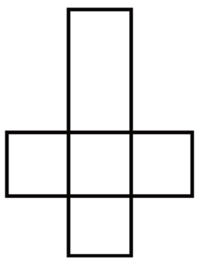

Figure 1. Two rectangles drawn in the plane.

However, instead of thinking of the rectangles as two 4-sided polygons in the plane, we will think of the rectangles we draw as giving rise to a graph in the dots/lines or vertex/edges framework (Figure 2). When two sides of different polygons in Figure 1 intersect we will consider that as a vertex of a graph and the original corners of the rectangles (Figure 1) will be regarded as vertices of our graph as well. From this perspective do you see how we can interpret Figure 1 as a graph drawn in the plane with 6 regions, 12 vertices, and 16 edges? When a vertex has k edges present (meeting at the vertex) we say it is k-valent or has degree k, and the number of k-valent vertices of a graph drawn in the plane (plane graph) will be denoted by $v(k)$. We can also count the number of sides of each region of the graph drawn in the plane so that edges meet only at vertices. Figure 1 shows 6 regions, 5 of them "bounded" and one unbounded (sometimes called the infinite face) region (face). Use $p(i)$ to denote the number of faces with i sides. For Figure 1 "recast" as a graph, we have $v(2) = 8$ and $v(4) = 4$, while $p(4) = 5$ and $p(12) = 1$. Recall that to help avoid errors when we do counts we can make some checks, which might help us see that a count is not correct. The number of vertices (V), faces (F), and edges (E) of a connected graph (a graph where for any pair of vertices one can walk along edges to get from one vertex to the other) obey the important relationship for connected plane graphs, often referred to as Euler's "polyhedral" formula

$$V + F - E = 2$$

because the vertices, faces, and edges of a (strictly) convex 3-dimensional polyhedron obey this equation. Since in plane graphs (without an edge joining a vertex to itself, a feature none of the graphs discussed here will have), each edge contributes a count of 1 at each of its two end points to the valences of the vertices at its endpoints, and each edge also contributes to the count of the number of sides in exactly two faces, we have:

$$\sum ip(i) = \sum kv(k) = 2E \quad (*)$$

where in the first sum i takes on all values of the different face sizes of the graph, and in the second sum k takes on all of the values of valence(degrees) that occur in the graph.

Thus the "data" for the graph associated with Figure 1, shown in an isomorphic (same structure) graph in Figure 2 (c), are:

$$V = 12,\ F = 6,\ E = 16$$

$v(2) = 8$ and $v(4) = 4$; $p(4) = 5$ and $p(12) = 1$. As a check using (*) we see $2(8) + 4(4) = 4(5) + 12(1) = 32 = 2E$. So E is 16, which can be verified by a direct count in Figure 2(c) (or Figure 1 interpreted as a graph).

What patterns do you see when you look at the objects in Figure 2?

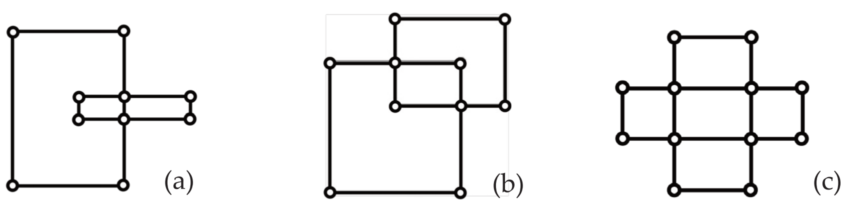

Figure 2. Example where two rectangles with their axes aligned intersect in "different ways." To highlight the differences we think of the intersecting rectangles as graphs (dots and lines kind) drawn in the plane. When two sides of a rectangle meet at a point we put a vertex at this crossing in addition to highlighting the original 4 vertices of each of the distinct original rectangles.

Each of the three diagrams in Figure 2 can be viewed as taking two rectangles with axes aligned for convenience in the horizontal and vertical directions and showing what happens when the rectangles intersect (overlap) with the condition that the rectangles do not touch at a single vertex or along an edge. We will not allow (though we could, perhaps getting interesting patterns in this case also) that the axis-aligned rectangles be such that one lies entirely inside another, as shown in Figure 3, since in this case the plane graph associated with the rectangles is not connected.

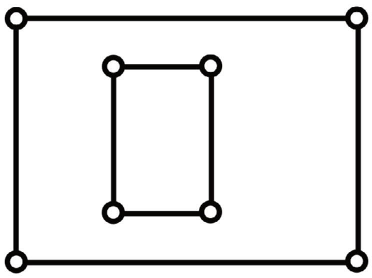

Figure 3. Two axis-aligned rectangles, one inside the other.

As graphs, the diagrams in Figure 1 have vertices of different valence (number of edges at a vertex) and faces with various numbers of sides, including the face that is unbounded. I will use $v(i)$ for the number of vertices of valence i and $p(i)$ for the number of faces with i sides.

The valence(degree) "data" for Figure 1 here are:

a. $v(2) = 8$; $v(4) = 2$;

b. $v(2) = 8$; $v(4) = 2$;

c. $v(2) = 8$; $v(4) = 4$.

Note that while the numbers for cases (a) and (b) are the same, the two graphs are not isomorphic (as maps in the plane—embedded graphs) because the sizes of the faces in these two graphs differ.

For those who like numerical patterns as well as geometric patterns, consider the plane graph shown in Figure 4.

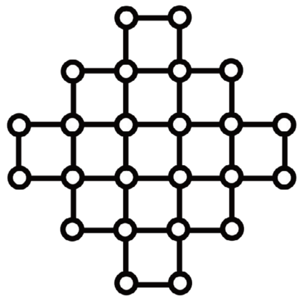

Figure 4. Three rectangles (one of which is a square) whose associated plane graph has bounded regions all of which are 4-sided.

Note that all of the bounded faces of this graph are 4-gons, and the infinite face is a 20-gon. Do you see how you can interpret this graph in the spirit of the discussion above as arising from three axis-aligned rectangles? You might enjoy looking for the graph that arises in this "family" for 4 rectangles and finding a formula for the number of "small" squares that occur. Hint: The sequence involved starts: 1, 5. The diagram in Figure 4 would represent the next graph in the sequence.

Investigation 1

Determine the sequence of number of 4-gons that can arise by using n rectangles to get patterns of cells such as that in Figure 4. Can you find other interesting things to count in the diagrams that "generalize" Figure 4?

Investigation 2

Count the number of inequivalent graph drawings that arise from axis-aligned plane rectangles when one uses n rectangles. (Note that two graphs can be inequivalent when they differ in their valences or face structure.) Given a positive integer s, we can consider ways that s can be written as the sum of a positive integer less than or equal to n. Thus, we speak of 7, 4 + 3, and 2 + 2 + 2 + 1 as partitions of 7. Given a set of s specific points in the plane one might raise questions as to whether the regions in an axis-aligned rectangle graph can "separate" the points into various partitions of s. When doing this we allow some of the faces of the graph associated with the partitioning rectangles to be empty, but you might consider other rules for "partitioning" the points involved. Thus, in Figure 5 we have a specific metric pattern of 8 points some of which lie on horizontal and vertical lines and perhaps there might be other triples of points of the set which lie on other lines.

Comment 1: When I use the term "rectangle" I allow for the possibility that some (all) of the rectangles might be squares. You might think about the question of what happens to some of your "counts" when all of the rectangles involved are squares. Perhaps some patterns involving the way "general" rectangles intersect cannot occur when all of the rectangles are squares!

Comment 2: None of the graphs in Figure 2 is the edge-vertex graph of a 3-dimensional convex polyhedron. It is of interest to consider how to add additional edges to such graphs so that the modified graphs arise as convex polyhedron in 3-space.

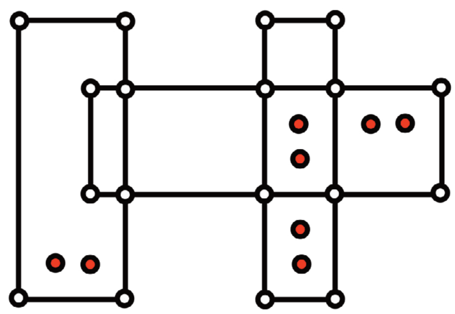

Figure 5. We start with the pattern of 8 red points and show that we can find a way of creating a pattern of axis-aligned rectangles so that the 8 points can give rise to the partition 2, 2, 2, 2 of 8.

Investigation 3

a. For the specific collection of 8 points shown in Figure 5, which partitions of the points are possible using 3 rectangles?

b. For 8 distinct points, arranged in the plane in a different way, study the different ways they can be partitioned using different numbers of rectangles.

c. For s distinct points (and you can vary the way these points are organized with respect to one another) and n axis-rectangles, study the way the regions formed by the n rectangles can partition the points subject to various "rules."

I hope you will find these questions intriguing to work on and think about! Happy researching!

References:

I have not provided references to encourage you to think about these problems in ways that may not have been considered in the past. However, there are research papers that deal with axis-aligned rectangles and related problems.

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Why a Column Encouraging Student Research?

There are many reasons mathematics is taught so extensively in grades K-12. These include the transmission of a body of knowledge in mathematics that was built up, literally, over thousands of years with contributions from all cultures (e.g. China, India, Arabia, Greece); the importance of mathematics in the workplace; the growing role that mathematics has for insight into so many areas of knowledge outside of mathematics — biology, economics, business, etc.

Current curriculum, pays so much time devoted to mathematical tools developed long in the past that students don't realize how much elementary mathematics is being discovered regularly. Mathematical methods improve and theorems stay theorems which creates a tension for the K-12 mathematics we should teach. One way to stretch student conceptions of mathematics is to show them examples of quick-starting questions that they can work on to have the sense of satisfaction in discovering new things for themselves and that are at the same time perhaps also new knowledge.

The items in this column will be drawn from graph theory, combinatorics, and other subjects which are not widely represented in the current K-12 curriculum but which illustrate that simple-to-state problems can serve to encourage students to try their hand at discovering new things and asking new questions about mathematical ideas. In some cases, it is not that nothing at all is known about the questions being posed but "references" to what is known are minimized so students will try new things that perhaps "experts" have overlooked. Since not all of the terminology used may be part of common knowledge, background know-ledge is available via a glossary:

https://www.york.cuny.edu/~malk/Glossary.html

Have fun and let us know if you make progress on any of these questions by sending an email with Student Research in the subject line to: info@comap.com