Folding Polyominoes to Boxes

Joseph Malkevitch

The collection of squares shown in Figure 1(a) can be cut out and folded on the line segments in the interior of the diagram into a $1 \times 1 \times 1$ cube or put differently, a $1 \times 1 \times 1$ box;. Figure 1(b) shows a drawing of a cube meant to suggest its 3-dimensional nature.

A polyomino is a collection of 1×1 squares in a plane that meet edge-to-edge and have no holes. Figure 1(a), (d) show examples and Figure 2 shows some additional examples. Figure 1(a) has 6 unit square cells and perimeter 14. Here, perimeter refers to the fact that viewed as a plane graph (a dots and lines drawing where edges meet only at vertices when drawn on a sheet of paper), Figure 1(a) a 14-gon. All the bounded faces (regions) are 4-gons (4 sides), there being 15 faces (regions, including one unbounded region) total. Disregarding the lines of a polyomino inside the unbounded region, one can think of the boundary of the polyomino as a rectilinear (orthogonal) polygon that might be convex or non-convex. Orthogonal polygons are plane polygons whose interior angles are 90°, 180°, or 270°. As a graph, the polyomino in Figure 1(a) has 14 vertices, 7 faces and 19 edges. These numbers obey Euler's formula for connected (one piece) graphs: V(vertices) + F(faces) – E(edges) = 2. Note that polyominoes have vertices of valence 2, 3, or 4 (sometimes called the degree of the vertices – the number of edges at a vertex). The 4-valent vertices of the graph are of two types: some are on the infinite (unbounded) face and some are in the "interior" of the simple circuit that forms the infinite face. In Figure 1(a) we have $v(2) = 7$, $v(3) = 4$ and $v(4) = 3$, where boundary 4-valent vertices are 3 in number and interior 4-valent vertices are 0 in number.

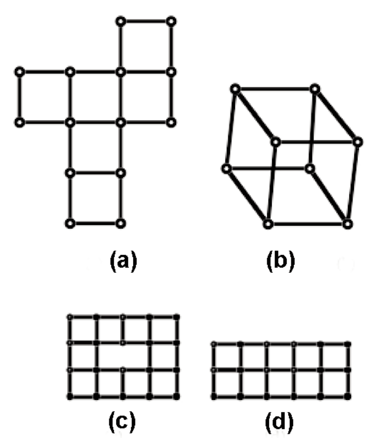

Figure 1. (a) A collection of squares that are a polyomino and will fold to a $1 \times 1 \times 1$ box–the cube. (b) Drawing of $1 \times 1 \times 1$ box, which suggests its 3-dimensional nature, but on a flat surface edges that don't meet in 3-space appear to meet. (c) A collection of 10 1×1 squares, but I will not allow a polyomino to have "holes," in this case a 1×2 rectangular hole. (d) A polyomino with 10 squares but which will not fold to a $1 \times 1 \times 2$ box even though the count of squares is correct, the problem in part being the presence of "internal" 4-valent vertices. The perimeter of this polyomino is 14.

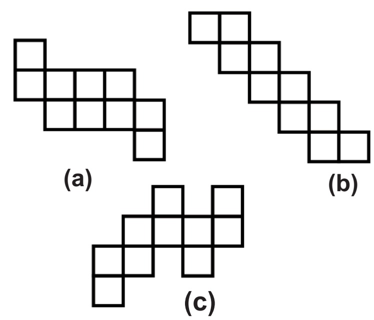

Figure 2. Polyominoes that fold to $1 \times 1 \times 2$ boxes. Two of these have perimeter 22 and one, perimeter 18.

It is well known that there are exactly 11 polyominoes (of the 35 polyominoes with exactly 6 squares) that fold to a $1 \times 1 \times 1$ box. By way of practice for what follows, try to find them all before looking at Figure 3. It turns out that if one cuts edges of a cube that form a spanning tree of the cube (a connected graph with no circuits that includes all of the vertices of the cube), then one can unfold the cube and flatten it out to one of the 11 polyominoes in Figure 3. Note that as trees, the edges that one cuts may be isomorphic (same structure) graphs and yet the polyominoes they unfold the cube to are different. Two polyominoes are considered the same (isomorphic) if they are congruent – one can cut out one of the two and place it on top of the other. It may not be possible to translate and rotate two congruent polyominoes drawn in the plane to one another because one might need to "turn over" one of the shapes before putting it on top of the other shape. In other words, one needs a "reflection" to transform one of the shapes into the other.

It is convenient to classify polyominoes from varying points of view. One such point of view is to count the number of cells in which vertical (horizontal) lines intersect the columns (rows) of the polyomino, as well as noting if these vertical (horizontal) lines cut the polyomino into connected segments. I will use the term vertically convex and horizontally convex to distinguish these cases. If a polyomino is both horizontally convex and vertically convex I will call it a convex polyomino. Some polyominoes are actually convex in the stronger sense that they are $m \times n$ rectangles. However, for folding polyomino questions, such polyominoes are of little interest since it is not difficult to see that they cannot fold to a box. When there is a run of cells in the vertical (horizontal) direction I will use the notation a^b to indicate this, and so one can code a polyomino by using two ordered strings, which I will separate by a semicolon, to indicate the "partition" structure of its cells. The entries in this notation will scan the polyomino from left to right and from top to bottom. One can extend the notation to code the cells that are not connected when a polyomino is not vertically or horizontally convex. One can also note the number of parts that the number of cells has, treating the cells that don't lie in a connected piece separately. Here are the "data" for the three examples in Figure 2. Figure 2(a) and 2(b) convex; Figure 2(c) vertically convex but not horizontally convex. The non-convex example in Figure 2 has 6 parts horizontally and 5 parts vertically.

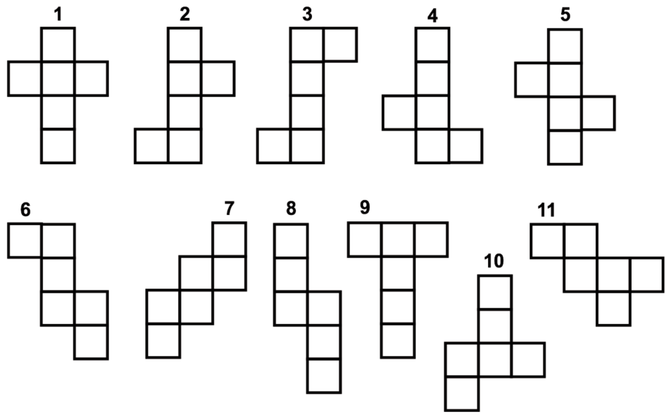

Figure 3. The 11 of 35 polyominoes with 6 squares that fold to a 1 × 1 × 1 box (cube).

1. 1,4,1 ; 1, 3, 1^2
2. 1,4,1 ; 1,2,1,2 Note one could write this $(1,2)^2$
3. 1,4,1 ; 2, 1^2, 2
4. 1,4,1 ; 1^2, 2^2
5. 1,4,1 ; 1, 2^2, 1
6. 1,3,2 : $(2,1)^2$
7. 2^3 ; 1,2^2,1
8. 3^2; 1^2,2,1^2
9. 1,4,1; 3, 1^3
10. 2,3,1; 1^2,3,1
11. 1,2^2,1; 2,3,1

If we take into account the parts only and not their order, then these codes do not uniquely determine the polyomino involved (e.g. 4 and 5). However, if the order matters, then these codes do distinguish all 11 cases, but it is not exactly clear how to draw the polyomino from its code.

Convince yourself that a polyomino that folds to an $a \times b \times c$ box consists of $2ab + 2ac + 2bc = 2(ab + ac + bc)$ squares. Thus, for a $1 \times 1 \times 2$ box folded from a polyomino one needs 10 squares, and for a $1 \times 2 \times 2$ box one needs 16 squares.

Warm up:

Verify that for values of k = 1, 2, 3, 4, there exists a polyomino with horizontal "code" all 2s that can be folded to a $1 \times 1 \times k$ box. (I don't know if this pattern continues.)

A very general question that is not well explored is:

Question 1:

What polyominoes fold into $a \times b \times c$ boxes?

However, even questions about the more special class of $1 \times 1 \times k$ boxes have many open and unexplored questions.

Question 2

1. What is the smallest perimeter and what is the largest perimeter of a polyomino that folds to:

a. $1 \times 1 \times k$ box

b. $a \times a \times b$ box

c. $a \times b \times c$ box

Question 3

Given an integer n that occurs as the number of squares for a convex $a \times b \times c$ box, for which partitions of n is there a polyomino that folds to this box?

Extension: If n is the number of squares that occurs for a polycube with n squares, for which partitions of n, say, in the horizontal direction is there a polyomino that folds to this polycube? Polycubes are the 3-dimensional analog of polyominoes.

The polycubes with 4 cubes are shown in Figure 4.

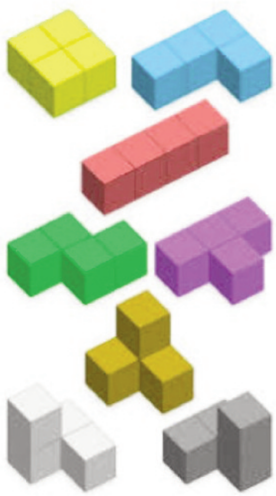

Figure 4. There are 7 incongruent polycubes with 4 cubes. The two bottom examples are congruent.

Two of these polycubes are boxes in the sense being used earlier ($1 \times 2 \times 2$ and $1 \times 1 \times 4$). Note the variation in the number of "surface" squares in these polycubes. Note that the two polycubes in the bottom row are congruent to each other using a reflection and so there are only 7 incongruent such polycubes.

One result that is probably implicit in what others have done, but it is hard to sample the research literature for what is known since much work is in the form of computer enumerations, is the following:

If B is a $1 \times 1 \times k$ box, then the box consists of $2(2k + 1)$ squares and there is a polyomino with partition $(2k + 1)^2$ having two rows of $2k + 1$ cells. The polyomino shown in Figure 5 folds to a box with these dimensions.

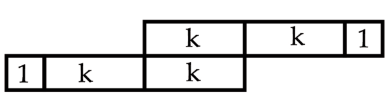

Figure 5. For a k at least 1 this polyomino will fold to a $1 \times 1 \times k$ box.

Note that polyominoes with numbers 7 and 8 (Figure 3) have the interesting property that they contain only one part. Given that we have 6 squares in these polyominoes and the only (proper) factors of 6 are 2 and 3, we potentially have that 3^2 and 2^3 codes might fold to a $1 \times 1 \times 1$ box and in fact there are polyominoes that work. Note that using combinatorics allows us to give an upper bound on how many 3^2 polyominoes there are. The first row of 3 squares can only be chosen in one way, and this row has three columns. The next row of squares can be lined up under one of three columns and thus there are at most three such incongruent polyominoes. We can make a similar count for an upper bound for the number of polyominoes with horizontal code 2^3. The first row can be chosen in two ways, and the next row can be chosen to line up under one of two cells in the previous row. Finally, for the third row, again, there are two ways to line up under the previous row, so using the fundamental principle of counting we have that there are at most $1 \times 2 \times 2 = 4$ different polyominoes that we can have with this code. Thus, in general we see that without "insight" of the kind we see in Figure 5 there are many cases to look through to try to see "constant" partitions (such as 3^2 and 2^3 for a $1 \times 1 \times 1$ box) that might fold to a $1 \times 1 \times k$ box when the polyomino has $2(2k + 1)$ squares.

Question 4

Let P be the set of perimeters that can occur when folding a $1 \times 1 \times k$ box. What can be said about this set of numbers?

General problem:

There are many numbers that can be associated with a polyomino: n, the number of square cells in the polyomino; p the perimeter of the infinite face of the drawing of the polyomino in the plane; the numbers that appear in a partition of n. Explore the space of these integer parameters for when one can for sure or can't for sure construct a polyomino with these parameters that folds to an $a \times b \times c$ box.

While there are many polyominoes that fold to boxes, combinatorially the graphs of these boxes are isomorphic to the graph of a cube, unless one adds vertices to the "polyhedron" showing how to subdivide its faces into squares. As a convex polyhedron an $a \times b \times c$ box has 8 vertices, 6 faces and 12 edges but many interesting folding questions emerge when one sub- divides the faces of a box into squares of equal size and think of them as arising from folding polyominoes.

Have fun trying these problems, developing new ones of your own, and learning the nifty mathematics related to folding polyominoes!

References

Tadaki, R. and K. Amine, Search for developments of a box having multiple ways of folding by SAT solver. https://arxiv.org/abs/2005.02645

O'Rourke, J., Folding Polygons to Convex Polyhedra, In National Council of Teachers of Mathematics (NCTM) 71st Yearbook: Understanding Geometry for a Changing World, Eds. Tim Craine, Rheta Rubenstein, pp. 77-87, 2009.

Joseph Malkevitch
Department of Mathematics and Computer Studies
York College (CUNY)
Jamaica, New York 11451