From Trees to Polyhedra

Joseph Malkevitch

Polyhedra have fascinated mankind since ancient times. Mathematical work about polyhedra has been done in Egypt, Greece, China and India. Typically, it is convex polyhedra that are discussed (3-dimensional surfaces bounded by planes, no notches or tunnels allowed). Historically, it has been the symmetry and metrical properties of convex polyhedra that have gotten a lot of attention. But it turns out that the dots and lines diagrams called graphs (Figure 1) can be used to get surprising rich insights into polyhedra.

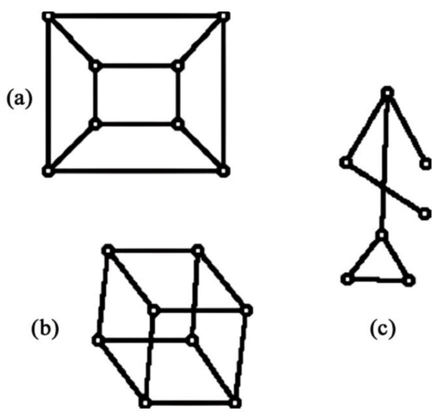


Figure 1 ((a) Shows a graph of the cube drawn in the plane. (b) Cube drawn so as to suggest its 3-dimensionality, but where edges cross at points that are not vertices. (c) Graph that is not the graph of a polyhedron or a tree.

Here I will look at a surprising way of constructing plane graphs that turn out to be the vertex-edge graphs of 3-dimensional polyhedra. These special polyhedral graphs have fascinating properties, some of which have been explored, but I don't feel properties of the graphs or of the related polyhedra are as fully understood as they might be. This sets the stage for many student research explorations. A major tool here is the theorem of Ernst Steinitz which reduces the study of 3-dimensional polyhedra (from a combinatorial point of view) to the study of a special class of graphs that have been drawn in the plane so edges meet only at vertices (see Figure 1 (a) or Figure 4 (b)). Such graphs are known as *plane* graphs. When a plane graph has the additional property that for any pair of its vertices *s* and *t* one can find at least three paths starting at *s* and ending at *t* that only have *s* and *t* in common, this property is called 3-connectedness. Since the graph in Figure 4 (b) is plane and 3-connected there is some convex 3-dimensional polyhedron that has this graph as the graph of its vertices and edges. Here we will restrict our attention to graphs that have been drawn in the plane with edges meeting only at vertices. The planar (isomorphic to graphs that can be drawn in the plane without crossings) and 3-connected graphs are known as 3-polytopal graphs.

The *valence* of a vertex of a graph is the number of edges at that vertex. Thus in Figure 1, the valences of the vertices shown in Figure 1 (c) are 3, 2, 1, 1, 3, 2, 2 scanning from top to bottom and left to right. The starting point for our investigation is the notion of a tree that has no 2-valent vertices. A graph is an example of a *tree* if it is connected, meaning one can walk between any pair of vertices along a path, and has no simple circuit. Having a simple circuit means one can start at some vertex and return to that vertex without traversing any edge or vertex (other than the start vertex) more than once. A *caterpillar* is a tree all of whose non-1-valent vertices lie on a path (see (Figure 2 (a) and (d)). It is possible for a graph to be a caterpillar with respect to two different paths, for example the paths from vertices 1 to 6 or from 7 to 9 in Figure 2 (d).

In Figure 2 all the graphs are trees, and Figure 2 (d) shows labels attached to the 9 vertices of the tree. Figure 2 (a) has a 2-valent vertex but the others have no 2-valent vertices.

Trees always have at least two vertices of valence 1 that are often described as being the *leaves* of the tree. Furthermore, trees have the property that between any two vertices there is exactly one path; also the number of vertices in a tree is always exactly one more than the number edges in the

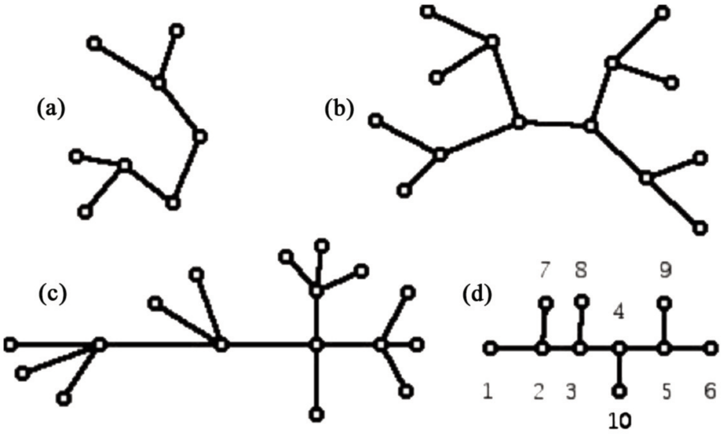


Figure 2 (a) A tree with two 2-valent vertices; (b) A tree all of whose vertices are leaves or 3-valent; (c) A tree whose vertices are leaves or 4-valent; (d) A labeled caterpillar with 6 vertices whose non-leaves are 3-valent.

tree. (This is a special case of the Euler Polyhedral Formula for connected graphs drawn in the plane: V (Vertices) + F (Faces) – E (Edges) = 2. Figure 2 shows some examples of the type of trees that are of interest here. Such trees can have all vertices of valence k (k at least 3) or a mixture of valences. The non-leaf vertices in Figure 2(b) and (d) are 3-valent and those in Figure 2(c) are 4-valent.

If $t(i)$ denotes the number of vertices of a tree of valence i, show that the values $t(i)$ obey the equation below. Often equations of this kind are shown using subscripts, t_1 rather than $t(1)$.

$$t(1) = 2 + \sum_{i=2}^{i=m} \big(t(i) - 2\big)$$

Again, Figure 3 shows a special kind of tree called a caterpillar. A caterpillar consists of a path (here, vertices 1, 2, 3, 4, 5, 6) with all of the other edges of the tree being edges that join leaves (1-valent vertices) to a vertex of this path. In Figure 3 there are 4 other such vertices attached to the path. Notice in this caterpillar all of the vertices are either leaves or have valence 3.

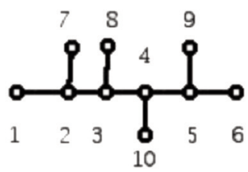


Figure 3 (Labeled caterpillar)

If one starts with a graph consisting of a tree without 2-valent vertices (called HIST trees) and passes a circuit through all of the 1-valent vertices of the graph, one gets a 3-polytopal graph. Such graphs are called Halin graphs (named for Rolf Halin) and they have many interesting properties.

By drawing examples you can verify that Halin graphs seem to have the following properties:

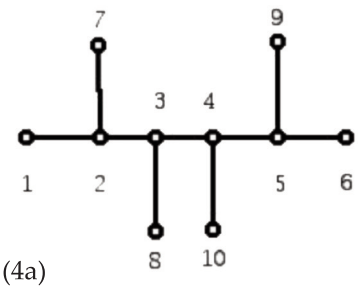
a. Any Halin graph admits a tour of the vertices that visits each vertex once and only once (such a tour is called a hamilton circuit (HC)).

b. Any Halin graph has circuits of vertices from lengths 3 up to the number of vertices in the graph, with the possible exception of a single circuit of even length. A graph with circuits of all lengths is called *pancyclic* and if it lacks a circuit of one length, almost pancyclic. The only situation where a single edge length circuit can be missing is when the tree on which the Halin graph is built has 3-valent vertices in addition to its leaves.

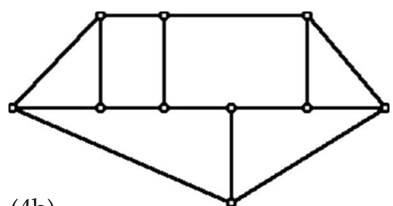
Figure 4 (b) shows an example where the starting tree is the tree consisting of 1-valent and 3-valent vertices shown in Figure 4 (a). It follows from $V + F - E = 2$ and 3-valence that if $p(k)$ denotes the number of regions (faces) with k sides, then the values of $p(k)$ obey:

$$3p(3) + 2p(4) + p(5) = 12 + \sum_{k=6}^{k=m} (k-6)p(k) \quad (*)$$

You should verify that for the labeled tree, in Figure 4 (a) passing a circuit through its leaves results in the 3-polytopal graph as in Figure 4 (b) (labels suppressed) and that for this graph there are circuits of all lengths from the 3 to 3 to 10 and the face lengths satisfy (*).

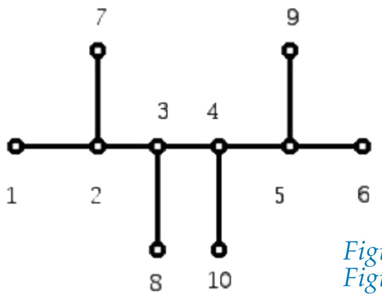


(4a)



(4b)

Figure 4(a) A labeled tree with no 2-valent vertices; b) A 3-polytopal graph created by passing a simple closed curve through the leaves in (a). One face in (b) is unbounded, and it has 6 sides.



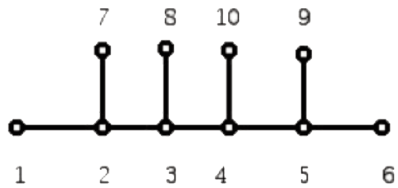


Figure 5 (Different embeddings of the tree in Figure 4(a) in the plane.)

Figure 5 shows two other ways to embed the tree in Figure 4(a) into the plane. Can you see that the Halin graphs obtained from these different drawings will not be isomorphic to the graph in Figure 4(b) or each other but still have face vectors that satisfy (*)?

Question 1:

1. What are the possible "face vectors" ($p(3)$, $p(4)$, ... , $p(\max)$) for polyhedra that are constructed from caterpillars where the valences of the caterpillar are:

a. 1-valent and 3-valent
b. 1-valent and 4-valent
c. 1-valent and 5-valent
d. 1-valent and mixtures of 3, 4, and 5-valent vertices.

Question 2::

For a fixed caterpillar tree with $t(3)$ and $t(1) = 2 + t(3)$, explore the different "face vectors" possible for different embeddings of the tree in the plane.

Question 3:

A partition of a positive integer s into at least two parts is a collection of at least two positive integers that can repeat (for us we need the integers to be each at least 3) whose sum is s. For s being 8, by way of example, we could use 3 + 5 or 4 + 4. Sometimes, when one passes circuits of lengths corresponding to the numbers in the partitions of s, the number of the leaves of a plane tree (or caterpillar) without 2-valent vertices, one gets a 3-polytopal graph. Thus, if a tree had 8 leaves, one could have circuits through 3 leaves and 5 leaves, or 4 leaves and 4 leaves, and these circuits could be chosen in different ways. Study the properties of the plane graphs obtained, in particular the 3-polytopes. In particular, study the face vectors that are possible, and the range of values for the number of sides of the unbounded face for the plane graphs that result. Such graphs would generalize the notion of a "Halin 3-polytope" when they are 3-connected.

Reference

Hom, Susan, Spanning Trees in 3-Polytopal Graphs, Doctoral Dissertation, Mathematics, City University of New York, 1993.

Joseph Malkevitch
Department of Mathematics and Computer Studies
York College (CUNY)
Jamaica, New York 11451

Why a Column Encouraging Student Research?

There are many reasons mathematics is taught so extensively in grades K-12. These include the transmission of a body of knowledge in mathematics that was built up, literally, over thousands of years with contributions from all cultures (e.g. China, India, Arabia, Greece); the importance of mathematics in the workplace; the growing role that mathematics has for insight into so many areas of knowledge outside of mathematics - biology, economics, business, etc.

Current curriculum, pays so much time devoted to mathematical tools developed long in the past that students don't realize how much elementary mathematics is being discovered regularly. Mathematical methods improve and theorems stay theorems which creates a tension for the K-12 mathematics we should teach. One way to stretch student conceptions of mathematics is to show them examples of quick-starting questions that they can work on to have the sense of satisfaction in discovering new things for themselves and that are at the same time perhaps also new knowledge.

The items in this column will be drawn from graph theory, combinatorics, and other subjects which are not widely represented in the current K-12 curriculum but which illustrate that simple-to-state problems can serve to encourage students to try their hand at discovering new things and asking new questions about mathematical ideas. In some cases, it is not that nothing at all is known about the questions being posed but "references" to what is known are minimized so students will try new things that perhaps "experts" have overlooked. Since not all of the terminology used may be part of common knowledge, background know-ledge is available via a glossary:

https://www.york.cuny.edu/~malk/Glossary.html

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