



# CONVEX ISOSCELES TRIANGLE POLYHEDRA

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The study of convex (no holes or notches) 3-dimensional polyhedra whose faces are isosceles triangles (two sides of one length, the third side of a different length) provides many new research problems suitable for high school and undergraduate students. These questions have their roots in mathematics whose investigation began over 2000 years ago! Before mentioning some specific research questions, let me put the questions in an historical context.

Euclid, in his *Elements*, provides a proof that there are five convex regular (sometimes called Platonic) solids. A convex polyhedron is called *regular* if all of its faces are congruent regular polygons (all sides the same length; all angles have equal measure), and all the solid angles at each vertex are congruent to each other. The solids involved are the regular tetrahedron, hexahedron (cube), octahedron, dodecahedron, and icosahedron. The number of faces of these solids is respectively, 4, 6, 8, 12, and 20. Archimedes (c. 287 – c. 212 BC) generalized this notation to require that all of the vertices be

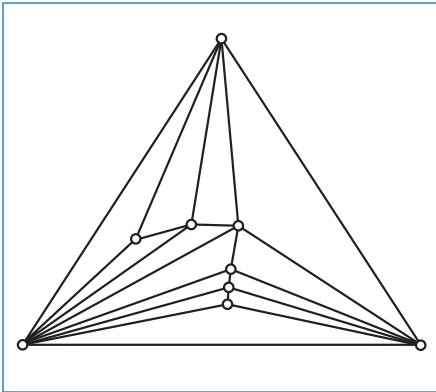
“alike” but allowed the faces to be regular polygons with different numbers of sides. The manuscript in which he did his work is now lost, but we are aware of his work because the mathematician Pappus (c. 290 – c. 350) describes what Archimedes discovered. Pappus describes 13 solids in detail. However, two infinite classes of solids that meet the definition that he gives go unmentioned. Furthermore, with hindsight we know that there are 14 solids (Archimedes, Pappus and Kepler missed the rhombcuboctahedron with one cap rotated) with regular polygonal faces and the same local pattern of faces around each vertex. The modern approach to the Archimedean solids, which requires that the symmetry group of each solid be able to move any vertex to any other, returns the count to the usual 13! Sometimes the Archimedean solids are called *semi-regular*. The “duals” of these solids (faces all alike; vertices can differ) were not studied until Catalan’s work of 1865.

In 1966, Norman W. Johnson published what he believed to be a complete list of 3-dimensional polyhedra, all of whose faces are regular polygons. This list includes the 5 regular solids, the 13 Archimedean solids, the prisms and anti-prisms (two infinite families)

and exactly 92 additional solids for which Johnson provided names in the spirit of the names for the Archimedean solids that were given by Kepler. However, Johnson did not provide a proof that his list was complete. This accomplishment was made by the Russian mathematician Viktor Abramovitch Zalgaller in 1969.

Embedded in what Johnson did is a count that was done independently of those convex polyhedra whose faces are equilateral triangles. Surprisingly, the idea of counting all inequivalent convex polyhedra whose faces are equilateral triangles is relatively recent. It turns out there are exactly 8 of these solids, sometime called the convex deltahedra. They have 4, 6, 8, 10, 12, 14, 16, and 20 faces. As a warm-up you may want to see if you can determine all of these polyhedra.

I will formulate the questions below in graph theory terms - using diagrams drawn in the plane that consist of dots (vertices) and straight line segments (edges) where all of the edges meet only at vertices. I will be interested in “combinatorial” as well as metrical (distance) questions. Figure 1 shows a triangulation of the plane where all the regions, including the unbounded region, are triangles. I will assume the triangulations have at least 4 vertices.



*Figure 1: A typical triangulation drawn in the plane. This one has 9 vertices and 14 faces, including the unbounded, infinite face.*

## General Questions

### Question 1

Which triangulations of the plane can be realized by convex polyhedra so that all of the faces are (strictly) isosceles triangles? (That is, one has a convex polyhedron whose edge-vertex graph has the same structure (is isomorphic) to the graph one starts with.)

### Question 2

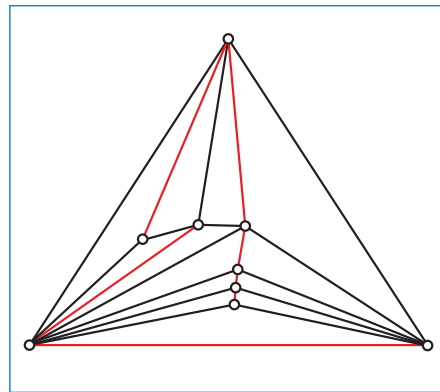
Which triangulations of the plane can be realized by convex polyhedra with congruent (strictly) isosceles triangles?

### Question 3

If one has two edge lengths  $b$  and  $r$  (not equal), which triangulations of the plane can be realized by convex polyhedra where there are equal numbers of (strictly) isosceles triangles of type  $b, b, r$  and  $r, r, b$ ?

For a better understanding of the issues the following discussions will provide a richer context. Given a triangulation such as that in Figure 1 we can try assigning “colors” to the different edges in the graph as a way of coding the lengths of the edges of the triangles in the

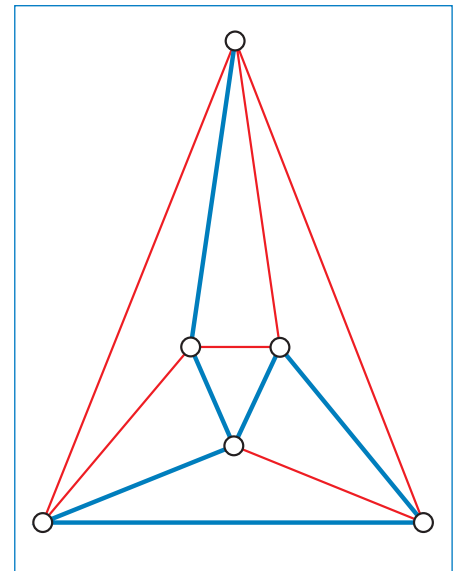
polyhedron that “realizes” the graph in 3-space. In this type of edge coloring I do not require that edges that meet at a vertex get different colors, which is the “traditional” way that edge colorings of graphs are defined. For my particular goal I have changed the “traditional” rules of assigning colors to the edges. When a coloring has been assigned so that all the triangles are  $b, b, r$ , I will call it a CIT coloring for a combinatorially (congruent) isosceles triangle coloring. It is known, and you might like to try to demonstrate for yourself, that a CIT coloring exists for any triangulation. When a coloring has been assigned so that there are equal numbers of  $b, b, r$  and  $b, r, r$ , triangles, I will call this a CIM, a combinatorially isosceles “mate” coloring where the number of mate triangles,  $b, b, r$  and  $b, r, r$  is equal.



*Figure 2: The triangulation in Figure 1 colored with two colors to represent different edge lengths. By using  $b$  and  $r$  as the colors, and having each triangle get two black edges and one red edge, if we can realize this coloring in 3-space with two edge lengths as indicated, all of the isosceles triangles would be congruent to each other.*

The actual colors we use does not matter. Figure 3 shows a CIM coloring of the regular octahedron with  $r$  (red) and  $b$  (blue) edges. Note that for a CIM coloring to exist, the total number of edges of the

triangulation must be even because by “symmetry” since  $b, b, r$  and  $b, r, r$  triangles are used in equal numbers, we need to have an even number of edges in the triangulation. This does not happen for all triangulations. A necessary condition for the existence of a CIM is that the number of faces (triangles) in the triangulation is divisible by 4. It is known, and you might like to try to demonstrate for yourself, that a CIM coloring exists for any triangulation.



*Figure 3: A triangulation where combinatorially there are four  $r, r, b$  triangles and four  $b, b, r$  triangles. This is a CIM coloring.*

Can you find other CIT colorings for the graph in Figure 4? (Think about the idea of the meaning that CIT colorings are “different.” Can you find CIM colorings of the graph in Figure 4? (Again, what would make CIM colorings different? Interchanging the roles of the two colors?)

While there are many ways to color the diagram in Figure 4 so that each face has two edges colored black and one edge colored red, it turns out to be impossible to realize this combinatorial type with all



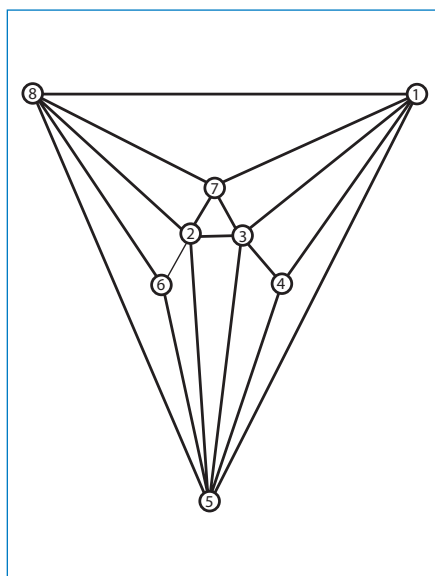
congruent isosceles triangles. It is not known if there is a way to construct convex polyhedra that realize any of the mate colorings for this particular graph. However, it is known that there are three infinite families of convex polyhedra with congruent isosceles triangle faces. Two are easy to find and the third less so. Can you describe these three infinite families?

#### Question 4

Other than the three known infinite families of congruent isosceles triangle convex polyhedra, are there other infinite families?

#### Question 5

Find mate colorings of the triangulation in Figure 4. Are there any convex 3-dimensional polyhedra which realize any of these colorings so that all the  $r, r, b$  triangles are congruent and all the  $b, b, r$  triangles are congruent?



**Figure 4:** A triangulation with 12 faces such that no CIT coloring of the edges can be realized in 3-space with congruent isosceles triangles.

#### Question 6

If it is true that every combinatorial type of triangulation with at least 4 vertices can be realized with isosceles triangles, is there some small number of edge lengths that this can be done with?

#### Comment:

Presumably it is easier to make isosceles triangle realizations of plane triangulations with more rather than fewer edge lengths.

#### Question 7

If a triangulation has  $F$  faces, what is the largest number of different edge lengths with which we can color the edges of its graph so that all the triangles are isosceles colored?

One can also study both colorings of triangulations where one uses only two edge lengths  $b$  and  $r$  but triangles can be colored  $b, b, b; r, r, r; b, b, r; b, r, r$ , as well as the realization of these colorings by convex polyhedra. Almost nothing is known about this family of polyhedra.

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