TETRAHEDRA

Tetrahedra, the analogue of triangles in the plane for three-dimensional Euclidean space, are surprisingly unstudied considering their "seeming" simplicity. To give you the flavor of the kinds of questions to be looked at, do you think you can construct a tetrahedron with edge lengths?

a. 7, 4, 4, 4, 4, 4 b. 18, 17, 16, 13, 13, 13

Throughout our discussion, lengths will refer to Euclidean distances. I will discuss the particularly interesting situation where potential edge lengths are positive integers; remember that the edges (sides) of a tetrahedron can have positive real numbers for lengths.

To continue it will help to give some background. Figure 1 shows a drawing of a tetrahedron in the plane that tries to convey the 3-dimensional quality of being a tetrahedron. Thought of as a graph, a tetrahedron is a geometric diagram consisting of dots called vertices, (the singular is) vertex) and line segment (edges). A tetrahedron has 4 vertices, 6 edges, and 4 faces

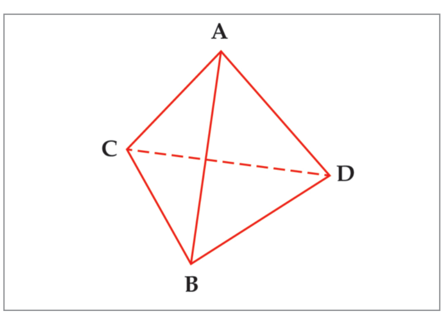

Figure 1 *(Hidden line drawing in the plane of a tetrahedron)*

For our purposes it is more convenient to use drawings such as that in Figure 2 for tetrahedra. This graph drawing has the same structure, i.e., is isomorphic to the graph in Figure 1. The four faces of the tetrahedron show up as three bounded regions and one "unbounded" region (infinite face). Figure 2 uses the letters a, b, c, d, e, and f to denote the edge lengths of the tetrahedron, using Euclidean distance. It is an example of a plane graph because the edges of the graph meet only at vertices. (The graph in Figure 1 is a planar graph because it is isomorphic to a plane graph.)

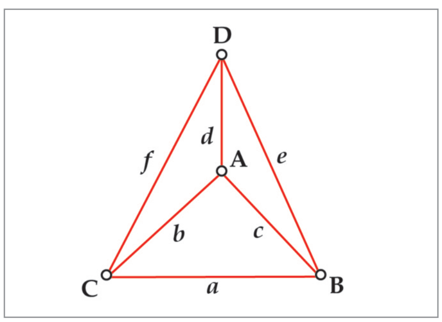

Figure 2 *(Schematic of a "potential" tetrahedron with edges of lengths a, b, c, d, e, and f.)*

Given a set of 6 positive numbers, we can ask if these 6 numbers used as the lengths of edges in Figure 2 can be "realized" by a tetrahedron in 3-dimensional space. However, there may be a set of numbers that can be realized but which, if one insists that the lengths be assigned to particular segments in Figure 2, the answer might be no! (Can you find an example?)

So when we ask if (17, 16, 13, 13, 13, 18) exists or not we are asking that in Figure 2, $a = 17$, $b = 16$, $c = 13$, $d = 13$, $e = 13$, and $f = 18$.

How does the analogous question work for triangles? We can ask if the sets {5, 4, 3} or {8, 4, 3} allow us to construct a triangle. The answer to this question is that as long as the sum of any two of the numbers in the set is greater than the third, a physical triangle can be made. If we specify an order for the three numbers, we can assign any of the three numbers to the three sides of the symbolic triangle ABC (see Figure 1) as long as the "triangle inequality" holds. Thus, all six triangles, for the set {3, 4, 5}, namely (5, 4, 3), (5, 3, 4), (4, 3, 5), (4, 5, 3), (3, 4, 5), and (3, 5, 4) exist. In fact for this particular triple the triangle we get will be a right triangle because we can use the converse of the Pythagorean Theorem, which like the Pythagorean Theorem itself is also true.

The situation for tetrahedra is much more subtle. If one assigns 6 numbers to edges in Figure 2 and for each of the 4 faces represented in that diagram there is a triangle with these lengths, it may not be true that there a 3-dimensional tetrahedron that realizes these 6 numbers as edge lengths of a tetrahedron. If there is actually a tetrahedron with 6 edge lengths it will have to have a positive volume. One can use determinants to find the volume of a tetrahedron in terms of its six edge lengths. The usual determinant for doing this is due to Arthur Cayley and Karl Menger (not to be confused with the economist Carl Menger, who was his father). This determinant is a 5 × 5 determinant, which means a lot of computation is involved. However, remarkably there is a 3 × 3 determinant that can be used instead. This is the McCrea determinant due to William McCrea (1904-1999), best known for his work in astronomy. The formula below appears in his book *Analytical Geometry of Three Dimensions* (reprinted Dover Press). It is worth noting that when the determinant is 0, a degenerate configuration that lies in the plane (complete quadrilateral results), that is, the 4 points involved in Figure 2 and the six distances can be chosen to lie in a plane—but there are subtleties here, too.

$$\begin{vmatrix} 2d^2 & d^2+e^2-c^2 & f^2+d^2-b^2 \\ d^2+e^2-c^2 & 2e^2 & e^2+f^2-a^2 \\ f^2+d^2-b^2 & e^2+f^2-a^2 & 2f^2 \end{vmatrix}$$

Figure 3 *(McCrea determinant to determine if a potential tetrahedron has positive volume. Not all collections of 6 positive numbers for which each subset of 4 obeys the triangle inequality determine a physical tetrahedron.)*

How many different tetrahedra are there for a set of {a, b, c, d, e, f} distinct numbers? The answer will depend on what we mean by different. Since the edge labeled a in Figure 1 can be given one of 6 values, the edge b can be assigned one of 5 values etc., the answer will be 720 different labelings where all 6 different numbers are used. (I will consider the issue of repeated edge lengths shortly). However, because the number of symmetries of the cube is 48 (24 rotational symmetries), no more than 30 of the 720 possible labelings can lead to tetrahedra that are not congruent ("different") from each other. In fact, the set of 6 integers {13, 14, 15, 16, 17, 18} can be realized by 30 different tetrahedra. This particular set of 6 has the "extra" property that if one uses a subset with repeats, such collections of 6 numbers also yield tetrahedra (e.g. 13, 13, 13, 18, 18, 18). As noted, there are some sets of 6 numbers where each subset of 4 numbers obeys the triangle inequality and yet there is no tetrahedron with these numbers.

Question 1

For which integers t from 0 to 30 is there a tetrahedron with exactly t inequivalent tetrahedra associated with a fixed sextuple S? (For example, can you find a set of 6 positive integers so that there are exactly 11 inequivalent tetrahedra with these edge lengths? Note that to achieve these different values some of the 6 integers may be equal, and it is of interest for fixed t, what values of s from 1 to 6 are possible.)

Work related to this problem exists. Simulations that show relative frequencies of the 31 values of t have been carried out, but there seems not to be explicit integer length examples for each value of t in the range 0 to 30.

Question 2

In the framework of Question 1, for each value of t, what partition types of tetrahedra arise that have exactly t realizations for a sextuple of positive integers which allows the realization of a tetrahedron?

Question 3

Explore the sextuple space of positive integers that can be realized by integers. An example of such an exploration would be, suppose sextuple S can be realized by a tetrahedron. Suppose 1 is added to each element of the sextuple, to get S'. Can S' always be realized? Similar questions arise if we subtract 1 from each entry.

Investigation:

Associated with a sextuple one can define a "dual" tetrahedron sextuple using the edge labels in Figure 2. For each edge in Figure 2 there is exactly one edge disjoint from this edge. Given $S = (a, b, c, d, e, f)$ (repeated lengths allowed) investigate the "pair" (S, S^*) where $S^* = (d, e, f, a, b, c)$.

References

Malkevitch, J., and D. Mussa, The transition from two dimensions to three dimensions - some geometry of the tetrahedron, *Consortium* 105, Fall/Winter 2013, p. 1-5.

Hildebrand, A., (Faculty mentor), The Broken Stick Problem in Higher Dimensions.

Page, Alex, Yuliya Semibratova, Yi Xuan, Eva Rui Zhang, M. Tip Phaovibul, and A. J. Hildebrand. "Calculus, Geometry, and Probability in *n* Dimensions: The Broken Stick Problem in Higher Dimensions IGL Project Report, Spring 2015." (2015).

https://faculty.math.illinois.edu/~hildebr/ugresearch/Hildebrand-Calculus-Spring2015-report.pdf

Wirth, K., and A.Dreiding, Edge lengths determining tetrahedrons, Elemente der Mathematik 64 (2009) 160-170.

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