CONSORTIUM 15
Student Research Corner

Drawings of Number Sequences

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Why a Column Encouraging Student Research?

There are many reasons mathematics is taught so extensively in grades K-12. These include the transmission of a body of knowledge in mathematics that was built up, literally, over thousands of years with contributions from all cultures (e.g. China, India, Arabia, Greece); the importance of mathematics in the workplace; the growing role that mathematics has for insight into so many areas of knowledge outside of mathematics - biology, economics, business, etc.

Current curriculum, pays so much time devoted to mathematical tools developed long in the past that students don't realize how much elementary mathematics is being discovered regularly. Mathematical methods improve and theorems stay theorems which creates a tension for the K-12 mathematics we should teach. One way to stretch student conceptions of mathematics is to show them examples of quick-starting questions that they can work on to have the sense of satisfaction in discovering new things for themselves and that are at the same time perhaps also new knowledge.

The items in this column will be drawn from graph theory, combinatorics, and other subjects which are not widely represented in the current K-12 curriculum but which illustrate that simple-to-state problems can serve to encourage students to try their hand at discovering new things and asking new questions about mathematical ideas. In some cases, it is not that nothing at all is known about the questions being posed but "references" to what is known are minimized so students will try new things that perhaps "experts" have overlooked. Since not all of the terminology used may be part of common knowledge, background know-ledge is available via a glossary:

https://www.york.cuny.edu/~malk/Glossary.html

Have fun and let us know if you make progress on any of these questions by sending an email with Student Research in the subject line to: **info@comap.com**

This research problem involves translating between geometric drawings and sequences of numbers associated with these drawings.

Figure 1 shows a connected plane graph with 8 vertices and 13 edges. A connected graph is one that consists of a single piece, and a plane graph is one that can be drawn in the plane so edges meet only at vertices (edges don't "cross"). Here I will be concerned with graphs that have no edge joining a vertex to itself (self-loop) and don't have two different edges which join the same pair of distinct vertices (multiple edges). The degree or valence of a vertex in a graph is the number of edges that appear at the vertex. Note that all of the bounded regions in Figure 1 are triangles - have 3 sides.

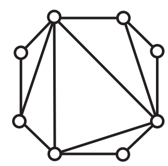

Figure 1 (A connected plane graph)

One can easily write down the degrees or valences of the vertices for the graph in Figure 1 in decreasing order and obtain the sequence of numbers: 5, 5, 4, 3, 3, 2, 2, 2

Inverse Problem

Given a collection of non-negative integers, which for convenience I list in decreasing order (Box 1 shows examples), is there a graph with these numbers as its degrees or valences? If such a graph exists, the number of its vertices will be equal to the number of items in the list. Here I will restrict my discussion to connected graphs though sometimes constructions leading up to a connected graph will not be connected. Using trial and error one can try to draw the graphs with the given sequences in Box 1.

Box 1: (Try to draw graphs which, if you are successful, have these numbers as degrees of their vertices.)

a. 4, 4, 4, 4, 4
b. 4, 4, 3, 3, 1, 1
c. 6, 6, 5, 4, 4, 4, 3, 2, 2, 2, 2, 1
d. 5, 5, 3, 3, 2, 2
e. 4, 3, 1, 1, 1, 1, 1

When you are successful, your graph may have some special properties. For example, there is a way to draw a graph for sequence (e) so that the result is a tree (connected and no circuits) while this can't be done for sequence (b), though you can find a drawing of this sequence. There is a graph that corresponds to the sequence (a), but no plane drawing of this graph is possible.

It is fun to try to draw a graph with a given set of degrees but for some such challenges it can be seen that the task cannot be done by invoking a fact about the degrees of a graph already known to Leonard Euler (1707-1783):

If G is a graph, the sum of the degrees of its vertices is an even number, which is equal to twice the number of edges of the graph.

□□□□□□□□□ *Any graph has an even number of vertices of odd valence.*

In light of this it is not possible to draw a graph with the degrees in Box 1(c)

A list of non-negative decreasing integers that arises as the valences of some graph is known as a degree or valence sequence. Figure 1 showed a special type of connected graph that arises from a convex polygon (this example being an 8-gon) that has been subdivided into triangles. Not all degree sequences have a drawing of this kind that gives rise to them.

Research question

Which degree sequences arise from triangulated convex □-gons drawn in the plane? (Find necessary and sufficient conditions for a degree sequence to arise from a triangulated convex □-gon.)

For example it is not difficult to see that, in addition to the previous example:

2, 2, 2 arises from a triangle; 3, 3, 2, 2 arises from a 4-gon; 4, 3, 3, 2, 2, arises from a 5-gon.

Though it may not be apparent, it is a fact that every triangulated plane convex polygon has at least two vertices of degree two, so this offers a necessary condition on a degree sequence to arise from a convex triangulated polygon.

We have developed the questions above with regard to triangulated convex polygons. Suppose one begins with a simple non-convex polygon. Are there degree sequences that can be obtained in this situation that can't also be obtained for triangulated convex polygons? Although it is not easy to prove it, every non-self-intersecting (simple) plane polygon can be decomposed into triangles.

Figure 2 shows that the degree sequence we initially looked at, 5, 5, 4, 3, 3, 2, 2, 2 can be realized by several graphs which are triangulated convex polygons that are not isomorphic (same structure). The two "inequivalent" triangulated polygons are shown side to side; you can check they have the same degree sequence.

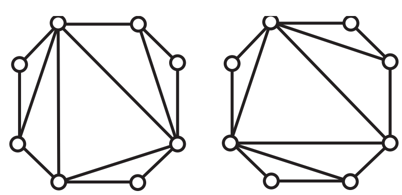

Figure 2 (Two inequivalent (non-isomorphic) triangulated 8-gons with the same degree sequence)

There are many theorems (facts) known about when sequences of integers arise from graphs. One appealing idea that may be useful for the unsolved problem above, is known as the Havel-Hakimi Theorem, which applies for graphs that are not necessarily connected. I won't state the theorem but give an example to help give you the idea:

Given a sequence such as the ones in Box 1, if a graph (not necessarily connected) can be drawn for a derived sequence, then there is a graph (not necessarily connected) that gives rise to the original sequence. (One can repeat this approach several times but care has to be taken that sometimes the order of the items in the derived sequence must be reordered.)

Example:

Original sequence: 5, 4, 3, 3, 3, 3, 2, 1
First derived sequence: 3, 2, 2, 2, 2, 2, 1
Second derived sequence: 1, 1, 1, 2, 2, 1
(Reordered sequence: 2, 2, 1, 1, 1, 1).

Since the graph at the top in Figure 3 has 2, 2, 1, 1, 1, 1 as its degree sequence, it is possible to "work backwards" as shown to draw a graph for the original sequence (bottom graph).

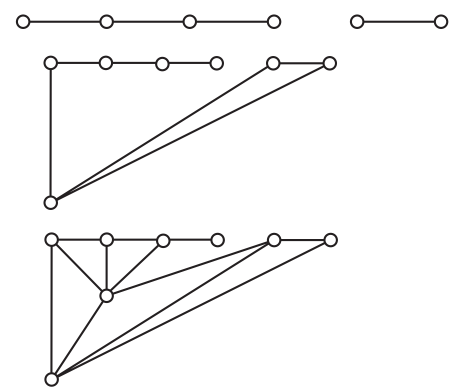

Figure 3 (Illustration of the Havel-Hakimi Theorem)

Have fun experimenting here, and perhaps you can make progress on the question of degree sequences arising from triangulated plane convex polygons.

Reference

D. West, Introduction to Graph Theory, (2nd. ed.), Prentice-Hall, Upper Saddle River, 2001.