Folding Clusters of Equilateral Triangles

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Why a Column Encouraging Student Research?

There are many reasons mathematics is taught so extensively in grades K-12. These include the transmission of a body of knowledge in mathematics that was built up, literally, over thousands of years with contributions from all cultures (e.g. China, India, Arabia, Greece); the importance of mathematics in the workplace; the growing role that mathematics has for insight into so many areas of knowledge outside of mathematics - biology, economics, business, etc.

Current curriculum, pays so much time devoted to mathematical tools developed long in the past that students don't realize how much elementary mathematics is being discovered regularly. Mathematical methods improve and theorems stay theorems which creates a tension for the K-12 mathematics we should teach. One way to stretch student conceptions of mathematics is to show them examples of quick-starting questions that they can work on to have the sense of satisfaction in discovering new things for themselves and that are at the same time perhaps also new knowledge.

The items in this column will be drawn from graph theory, combinatorics, and other subjects which are not widely represented in the current K-12 curriculum but which illustrate that simple-to-state problems can serve to encourage students to try their hand at discovering new things and asking new questions about mathematical ideas. In some cases, it is not that nothing at all is known about the questions being posed but "references" to what is known are minimized so students will try new things that perhaps "experts" have overlooked. Since not all of the terminology used may be part of common knowledge, background know-ledge is available via a glossary:

https://www.york.cuny.edu/~malk/Glossary.html

Have fun and let us know if you make progress on any of these questions by sending an email with Student Research in the subject line to: **info@comap.com**

Clusters of equilateral triangles assembled edge-to-edge (without holes) are called polyiamonds. Figure 1 shows two polyiamonds, each having 12 triangles, one of which is convex (no holes or notches) and the other is not convex.

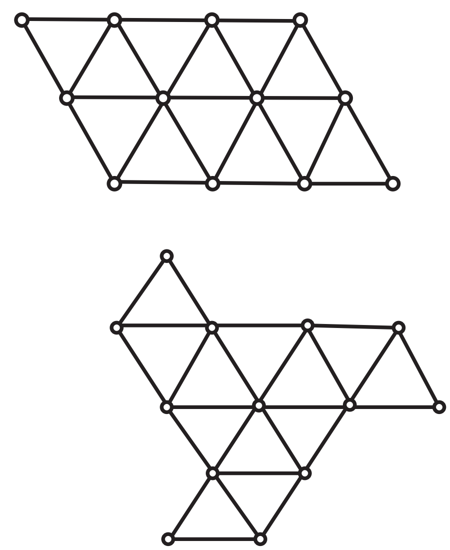

***Figure 1**: Two 12-triangle polyiamonds; one convex (top), one not convex (bottom).*

Counting the number of inequivalent polyiamonds with n triangles is not easy but it is known that the number of polyiamonds with 12 triangles is 3226 (3344 if holes are allowed). Both of these numbers seem surprisingly large to me.

Using the "internal" edges of a polyiamond as fold lines one can fold some polyiamonds into convex and/or non-convex 3-dimensional polyhedra. For example, the 4-triangle polyiamond in Figure 2 (a) can be folded to the regular tetrahedron by folding along the internal edges. Now one can take this tetrahedron and cut it along some of its edges and unfold the tetrahedron to a different 4-triangle polyiamond (Figure 2 (b)). Figure 2 (b) is one of a family of appealing polyiamonds which consist of m rows and n columns. Figure 2(b) is the 1 × 4 member of this family and the lower polyiamond in Figure 1 is the 2 × 6 member of this family.

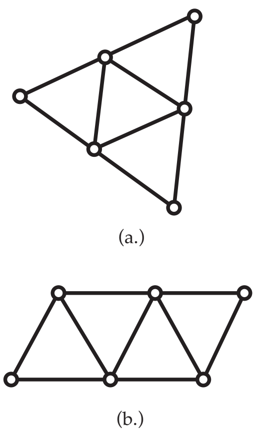

Figure 2: *Two polyiamonds that fold to the regular tetrahedron.*

There may be several different polyiamonds which will fold to the same 3-dimensional solid (I don't allow solids such as two tetrahedra pasted along a single edge), and the same polyhedron may unfold to many different polyiamonds. Some polyiamonds will not fold to any convex or non-convex 3-dimensional polyhedron, for example, the polyiamond in Figure 3.

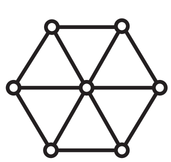

Figure 3: *A polyiamond that will not fold to a polyhedron.*

Fact

There are exactly 8 convex poyhedra whose faces are equilateral triangles, which are sometimes called the convex deltahedra, though some scholars allow convex deltahedra to include polyhedra whose faces are unions of equilateral triangles (see Figure 4).

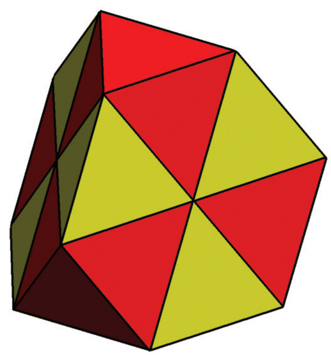

Figure 4. *A convex polyhedron whose faces are the union of equilateral triangles.*

Questions

0. What convex and non-convex polyhedra can be folded from the two polyiamonds in Figure 1?

1. What solids, deltahedra, union of equilateral triangle polyhedra which are convex and non-convex can be folded from: members of the $1 \times m$ family? from the $2 \times m$ family, and more generally the $m \times n$ family of polyiamonds?

2. Amazingly, it is still not known if for any convex 3-dimensional polyhedron, one can cut along the edges of the polyhedron and unfold the surface to a flat polygon (net). If the polyhedron has V vertices one would need to cut along $(V - 1)$ edges. This question is due to the late (2016) British mathematician Geoffrey Colin Shephard (1927-2016), to whose memory this note is dedicated.

As you explore the phenomenon of folding polyiamonds to convex and non-convex polyhedra, also be sure to write down new ideas and new questions that occur to you. The goal here is to experience what it is like working on problems that you and perhaps no one knows the answer to. Even if you make no progress in solving any of these you can enjoy the experience itself. Mathematics is more than the transmission of knowledge that others have conveyed to us. It can be a new window for viewing how to experience and learn about the world.

Reference

Shephard, G. C. (2013). Regular Polyhedral Clones. The Mathematical Gazette, 97 (540), 421-429.

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