2-D versus 3-D Style Drawings of 3-Dimensional Polyhedra in the Plane

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Polyhedra are an important topic of study by artists, model builders, scientists and mathematicians. Being a topic in geometry it is common to draw images of polyhedra as diagrams on a flat piece of paper, even though the "natural" home for polyhedra is in 3-dimensional space. Initially the study of such polyhedra was confined to convex polyhedra in 3-dimensional space but today anything goes - one can think about non-convex polyhedra in 5-dimensional space.

Here my goal is to look at the tradeoff between drawings on a flat surface of convex-3-dimensional polyhedra so as to appear "realistically" 3-dimensional and drawing the polyhedra as the 3-connected plane graphs (diagrams with dots and lines), which in many ways shows the mathematical structure of the polyhedron more accurately. We know by Steinitz's Theorem that the collection of planar 3-connected graphs (no loops or multiple edges) are exactly the vertex-edge graphs that can arise from the collection of bounded convex 3-dimensional polyhedra. Note: there is also the issue of how to interpret what one sees when one looks at a photograph, printed on flat paper, of a convex 3-dimensional polyhedron.

To avoid lots of definitions and to avoid being formal I will try to make my points with the diagrams themselves. A good place to begin are the diagrams, point-line graphs shown in Figure 1.

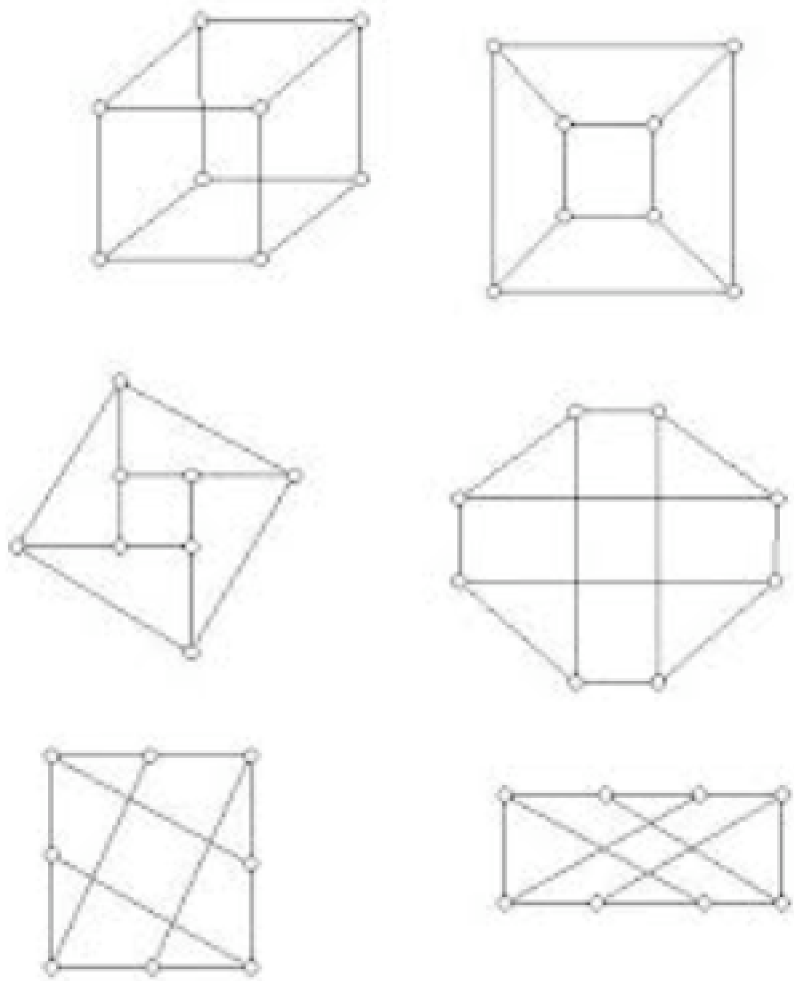


Figure 1 (Six different but isomorphic (same structure) as graphs ways to draw the graph of the 3-dimensional cube in the plane. The graph has 8 vertices, some of the lines representing the 12 edges intersect in points which are not vertices of the graph, which are shown here as small circles.)

In graph theory terms all 6 of these graphs are isomorphic, that is they have the same structure, and in terms of polyhedra they all are the graph of the combinatorial 3-dimensional cube - 8 vertices, 6 faces, and 12 edges, with each face a 4-gon and all of the vertices have three edges at a vertex. Remember that when a graph is drawn in the plane so that edges cross at points which are not vertices of the graph, then one can't talk about the "faces" (regions) of such a graph drawing. While all of these drawings show "regions," when the graph is not a plane drawing, the regions do not "correspond" to the faces of a polyhedron that the graphs shown here might represent. The drawings differ in a number of respects. The upper left diagram in Figure 1 attempts to convey the 3-dimensional quality of the way we often see cubes in the real world but the drawing "distorts reality" because cubes have 8 vertices, the small circles in Figure 1, but the edges in this drawing of the cube has points where edges meet by "accident" because of the way the drawing was done. Sometimes in such a drawing the edges that are "hidden" from view are not shown or are shown with a different thickness or are dotted to distinguish these "hidden" edges from the ones that one can see. The graph in the upper right hand corner has no accidental crossing but it lacks the 3-dimensional quality that we might find appearing from the top row left graph. However, with practice one can often develop intuition about what the polytopes guaranteed by Steinitz's Theorem might look like. The second row graph on the left is a plane graph but the 3-dimensionality is lost and while there are two regions which show that two of the faces have 4 sides, there are four other regions representing the faces which might appear to be "triangles" because in the drawing they are not represented by strictly convex polygons. Note that the other diagrams are interesting as graphs but don't help easily with thinking about these graphs as graphs of the 3-dimensional cube. Also some of these drawings seem to have rotational or reflection symmetries. However, it is not clear that all or some ways to realize these graphs in 3-dimensional also have rotational or reflection symmetries. Since all of the diagrams are isomorphic as graphs, they have the same symmetries as graphs but as drawings in the plane they don't all have identical symmetry.

If one makes a physical model of a convex polyhedron with membrane faces which is very symmetric, when one looks at this object in 3-space it is often difficult to tell what is "hidden" because one can't see the edges, vertices, and faces that are "hidden" in the back. Photos of such models sometimes don't make clear what one is seeing so perhaps it might help to have the photo sent with a hidden drawing of the model in the plane or a planar 3-connected version of the graph of the model.

To further illustrate some of the issues here, consider the diagram in Figure 2.

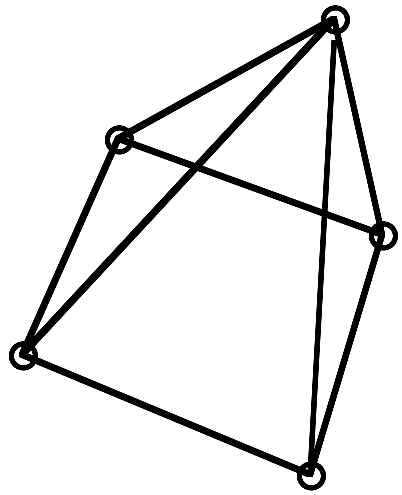
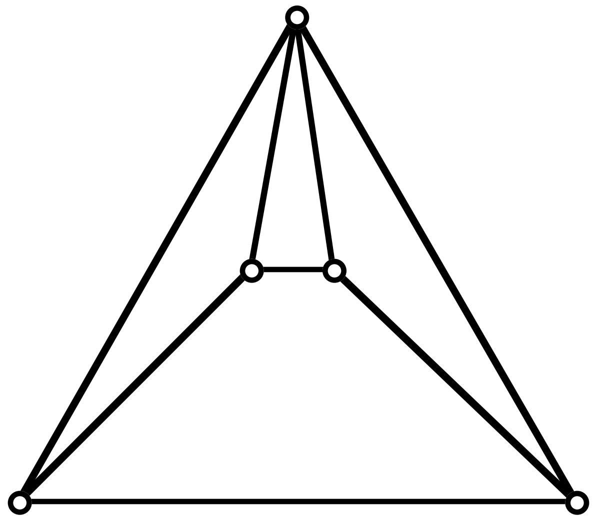


Figure 2 (A drawing in the plane of a "square pyramid" which tries to suggest some 3-dimensionality. This diagram could be improved to suggest that the base of the pyramid is a square and the triangles equilateral triangles. One could also have a less symmetric drawing in mind where the base is a rectangle. In many cases all that is of interest is the combinatorial type of the polyhedron being represented.)

This diagram is designed to suggest (but the lengths of the lines are not "accurate") a 3-dimensional pyramid where the base is a square and the triangles are 4 equilateral triangles, but how would one draw this diagram differently if the diagram was supposed to have the "base" be a rectangle rather than a square? For a base which is a rectangle, the solid could not have 4 equilateral triangles as the triangular faces. For any 3-dimensional polyhedron the polyhedron has an associated graph of its vertices and edges and this graph determines the "combinatorial type" of the polyhedron, but there are infinitely many different metric realizations of polyhedra for a given combinatorial type, many of which may be symmetric but one can always draw a version which has "no" symmetry.

Figure 3 shows two ways to draw the combinatorial type of the polyhedron shown in Figure 2 as a plane graph (drawn in the plane so crossings only occur at vertices). A theorem of Hassler Whitney (1907-1989) states that all the embeddings of plane 3-connected graphs are essentially the same. To the extent that such a drawing appears different it is because the choice for the number of sides of the unbounded face can differ in different drawings in the plane, even though the graphs they represent are isomorphic.



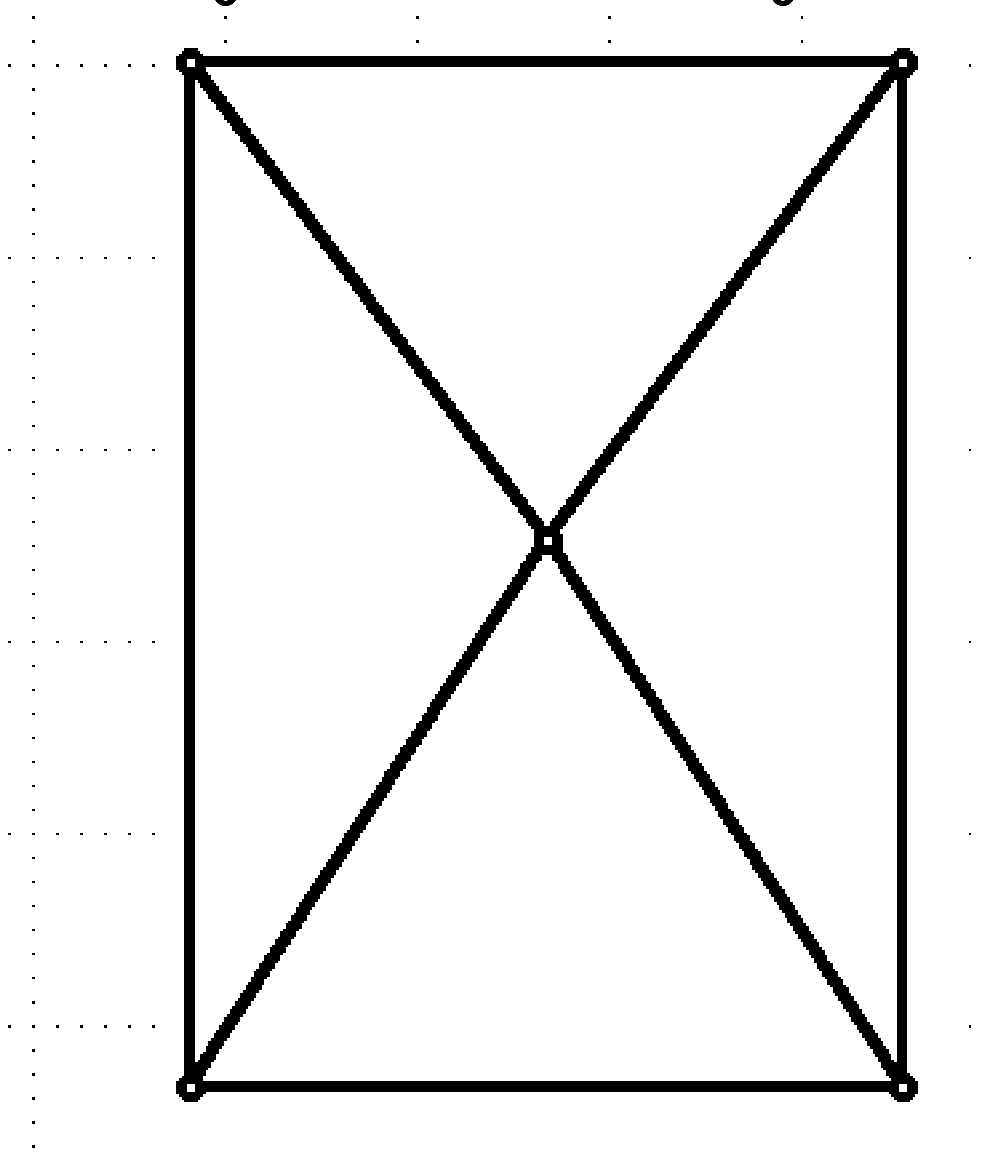


Figure 3 (The top diagram has the infinite or unbounded face a 3-gon (triangle) and the bottom diagram has the infinite or unbounded face a 4-gon. In both diagrams there are 4 regions with three sides and one region with four sides.)

If you are wondering what can go wrong when a plane graph is not 3-connected you can look at the two different embeddings of the graph shown in Figure 4. Though these two graphs are isomorphic one sees that the face sizes of the two different embeddings (drawings) in the plane are different. The numbers in the regions are the number of sides of the different faces involved. Note that the numbers in each case add up to 20.

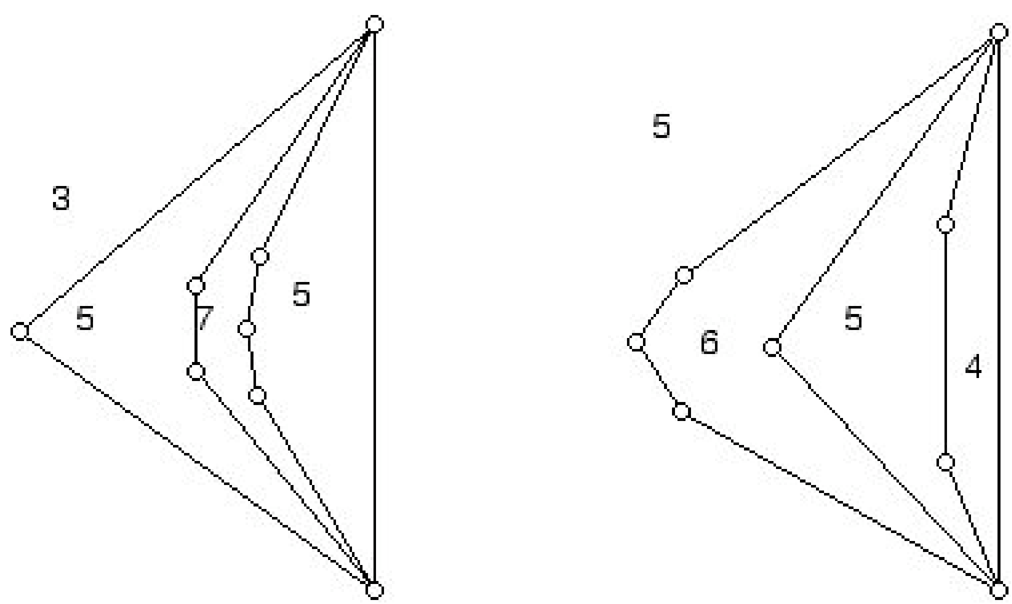


Figure 4 (Though the two graphs above are isomorphic they are not 3-connected. As you can see the drawings differ in the "face" structure they show. One drawing has a 7-gon and the other does not. The numbers indicate the number sides of the faces in the drawings in the plane.)

You have probably seen drawings of graphs in the plane which use curved lines (sometimes arcs of circles) to represent the edges. However, for graphs without multiple edges (more than one edge between the same pair of vertices) one can always avoid using curved lines. This result is sometimes called Fary's Theorem (Istvan Fary (1922-1984), though it has been independentl rediscovered several times. Also the straight line drawings of 3-polytopal graphs that one commonly sees often have strictly convex polygons representing the regions, and the interior of the edges that bound the infinite region is usually drawn as a convex polygon. This is because such a drawing is always possible - again this was rediscovered several times but often it is attributed to the great graph theorist William T. Tutte (1917-2002).

Given a plane 3-polytopal graph G, how symmetrical a version of this graph can be realized as a polytope in 2-space? Remember that there are infinitely many metrical versions of polytopes (all isomorphic to each other) which have G as their vertex-edge graph. The answer requires a bit of knowledge of group theory (from abstract algebra) and is given in a theorem due to the Swiss mathematician Peter Mani.

Given any graph H, all of the transformations of G onto itself, form an algebraic structure called a group. The number of elements in the group is a measure of how symmetric the graph is. If the size of the group is 1, then the graph has "no symmetry." The symmetries of a graph is often called the automorphism group of the graph.

Theorem (Peter Mani)

If H is a planar 3-connected graph whose group of symmetries (automorphism group of H) is G, then H can be realized in 3-dimensional space by a convex polyhedron P in such a manner that the symmetries of P (the isometry group of P) form a group isomorphic to G!

Mani's theorem says that in going from the symmetries of a 3-polytopal graph there is SOME realization of the graph as a polyhedron where no symmetries are "lost" in going from 2-dimensions to 3 dimensions. However, some have overstated what Mani's Theorem says. It is tempting to believe that if H* is a subgroup of the automorphism group H of a 3-polytopal graph, then there is a way to realize H as a polyhedron P* so that the symmetry group of P* is H*. This "extension" of Mani's Theorem is not true, even though it might seem "easier" to realize a "smaller" group than a larger one, and since the full group of symmetries of H can be realized by a polyhedron P* it seems that there should be another polyhedron P' which realizes the "smaller" collection of symmetries. However, things sometimes that seem reasonable are false.

As an example of the situation, consider the graph of the regular cube. It has 48 automorphism some of which can be thought of as "rotations" and some of which can be thought of as reflections. The rotation subgroup of the graph of the 3-cube has 24 elements whereas the automorphism group of the graph of the 3-cube has 48 symmetries. While all of the different drawings in Figure 1 are 3-cubes, their automorphism group as graphs has order 48. These graphs can be realized by the polyhedron we call the regular cube. There are infinitely many metric versions of this cube and the parameter they vary by is the edge length of the cube. While two cubes with different edge lengths are not congruent, they are similar in the sense that any such cube is a "scaled" of any other cube. All of these cubes have 48 isometries. However, there are polyhedra that realize the graph of the 3-cube with many fewer symmetries. In particular one can make such a polyhedron with no symmetries, or one with fewer than 48 symmetries. Thus, when one truncates a square pyramid with a square base and equilateral triangle faces, one gets the truncated pyramid whose graph is isomorphic to the graph of the regular 3-cube but such a polyhedron does not have 48 isometries, only 4 symmetries. This is the same number of symmetries as the square pyramid one started with. However, it is tempting to believe that since a regular cube has 48 isometries including reflections and rotations, there is a polyhedron of the combinatorial type of the cube (truncated pyramid) which has 24 symmetries. In fact, there is no polyhedron in 3-space which is of the combinatorial type of the cube which has exactly 24 isometries. Two non-isomorphic 3-dimensional convex polyhedra can have the same number of automorphisms for their graphs. Thus, the cube and the regular octahedron which realizes the graph in Figure 5 both have isometry groups with 48 elements.

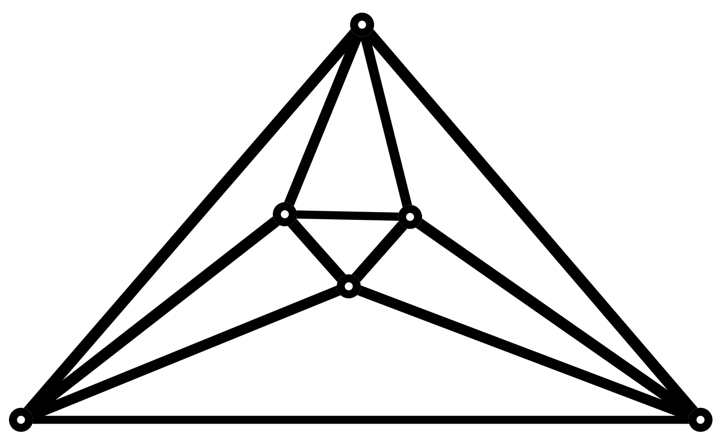


Figure 5 (Graph of the regular octahedron; all faces are triangles and every vertex has 4 edges at a vertex, that is, the graph is 4-valent (degree 4).)

The two other very useful tools in working with combinatorial properties of 3-dimensional convex polyhedra are Euler's Polyhedral Formula and Steinitz's Theorem.

Euler's Polyhedral Formula

If G is a connected plane graph then V + F - E = 2.

Here connected means that given any pair of vertices u and v in G one can move along edges of G to get from u to v. V stands for the number of vertices of the graph, F for the number of faces of the graph, and E for the number of edges of the graph. For example, you can verify that in Figure 5, where there are 6 vertices, 8 faces, and 12 edges that Euler's Polyhedral Formula is satisfied.

Comment:

It is remarkable that it appears that this simple relationship was observed for the first time in the 18th century rather than much earlier. It is not difficult to see that the graph of convex 3-dimensional polyhedron has a plane graph drawing.

Steinitz's Theorem

A graph is the edge-vertex graph of a 3-dimensional convex polyhedron if and only if the graph is 3-connected and admits a plane drawing (is a planar graph).

A graph H is 3-connected if given any two vertices of H, u and v, there are three ways (paths) to move from u to v along edges of the graph so that the only vertices in common on the "paths" that are used to get from u to v are u and v.

There are a variety of equivalent plane drawings of the "same" plane 3-polytopal graph and it does take some practice to visualize the polytopes associated with the plane graph one draws. However, it does have the advantage that when one counts vertices, edges and faces in such graphs these correspond "cleanly" to the number of vertices, edges and faces of the polytope.

The following example (Figure 6) may also help to clarify some of the issues. This polyhedron has an appealingly symmetrical realization in 3-space with 8 isosceles triangles that meet at the one 8-valent vertex, and the remaining 9 faces are all either equilateral triangles or squares. One way to think of this polyhedron is that it can be cut by a plane which intersects the solid in 8 vertices cutting it into TWO convex pieces (including the 8-gon lying in the cutting plane) each of which is a quite "symmetric" part. The unbounded face in Figure 8 corresponds to one of the 8 congruent isosceles triangles in this quite "symmetrical" realization. However if one constructs a physical model of this polyhedron there are ways to view this polyhedron so that all that one "sees" is:

a. What appears to be a polyhedron with an regular octahedron face, and the other faces are equilateral triangles and squares (5 squares and 4 equilateral triangles (see Figure 9) for a plane 3-connected version of this polyhedron)

b. What appears to be an a polyhedron which is a octahedral pyramid - its base is a regular octagon and its other faces are 8 congruent isosceles triangle faces. (No diagram provided. Use your knowledge of pyramids to visualize this other solid which has 9 faces, 8 triangles and one octagon.)

The point here is there are ways of viewing the solid in Figure 6 in 3-space that don't make clear what solid it is we are looking at. The planar 3-connected graph in Figure 6 has its downsides but it makes clear the combinatorial type of the solid we are looking at. Note that there are many convex 3-dimensional polyhedra that do not decompose into parts in this way - plane passing through vertices of the polyhedron that all lie in a plane so that both parts are convex polyhedra.

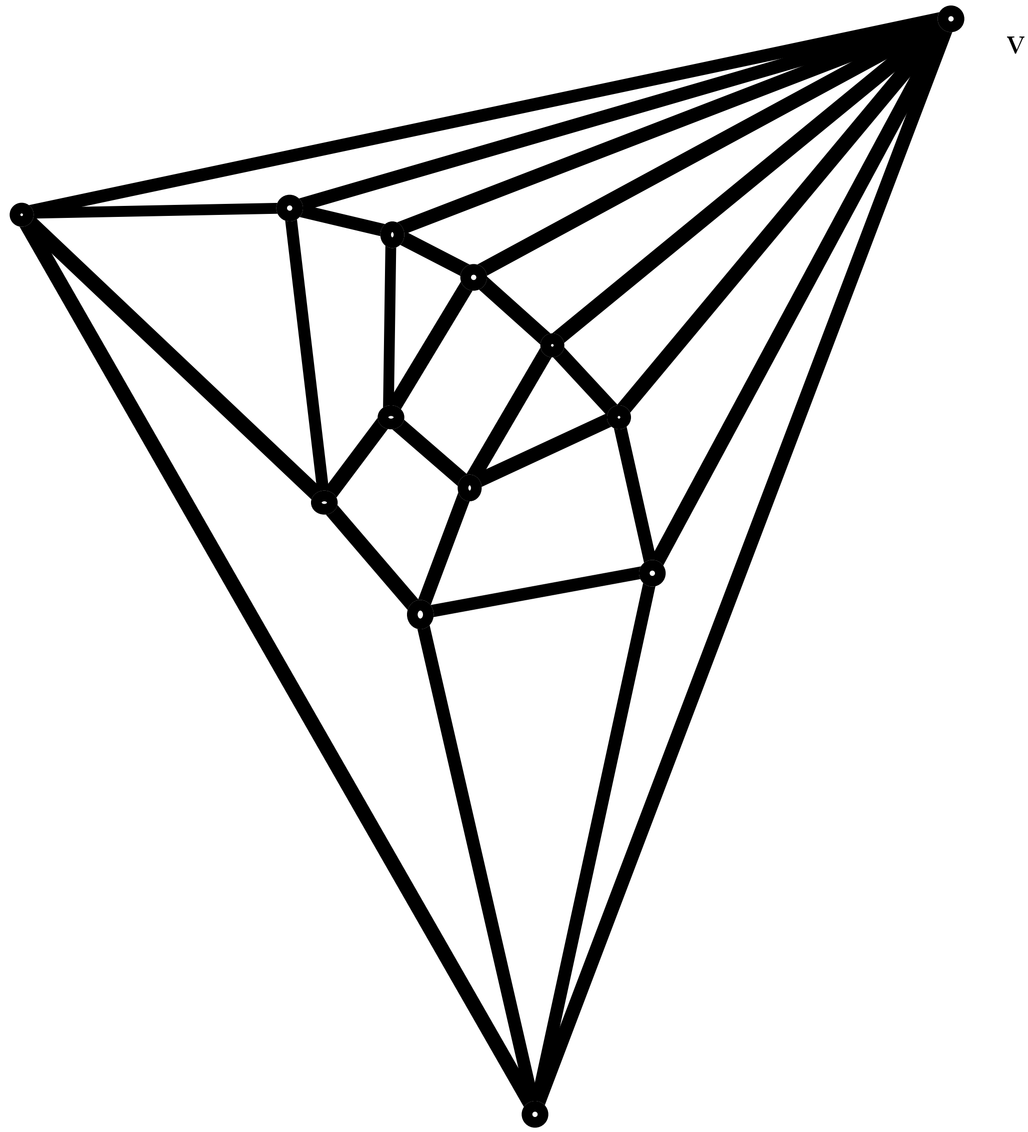


Figure 6 (A graph which can be realized in 3-space so that 8 of its vertices lie in one plane and which splits the polyhedron into two symmetrical convex polyhedra. One of these pieces will be an 8-gonal pyramid with isosceles triangle faces and a regular octagon as a base.)

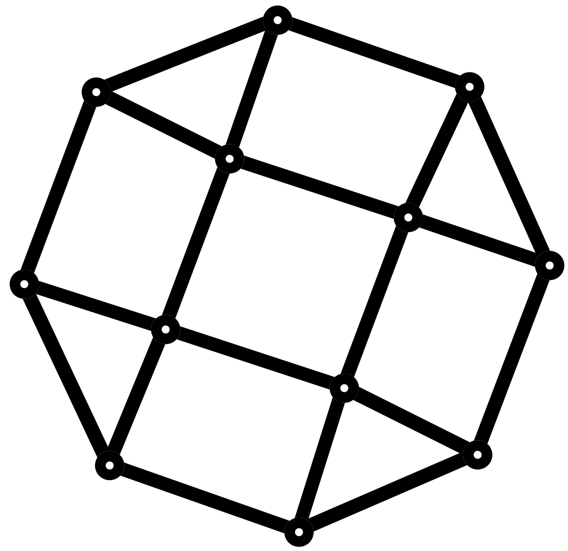


Figure 7 (A realization of the graph in Figure 6 can be split to realize this graph as a convex 3-dimensional polyhedron with a regular octagon, 5 squares and 4 equilateral triangles as faces.)

Now let us explore another "wrinkle" of dealing with 3-dimensional polyhedra and their drawings popularized by (perhaps discovered by him) Albrecht Dürer (1471-1528) who drew diagrams of convex polyhedra in the plane that arise from cutting edges of a connected graph with no circuits (tree) which includes all of the vertices of the polyhedron (spanning graph). So, one cuts all of the edges of a spanning tree of the original polyhedron. For the 1x1x1 cube this can be done in 11 different ways so as to flatten out the faces of the cube to lie flat in the plane. Sometimes such unfoldings are called nets. Note that there might be different ways of identifying the edges of the unfolding which will fold back to a different polyhedron, though this phenomenon does not occur for the 1x1x1 cube. Note also that isomorphic spanning trees as graphs can result in non-isomorphic nets. Furthermore, each net as a graph is not the edge-vertex graph of the cube or any convex 3-dimensional polyhedron because these graphs are not 3-connected (though they are planar).

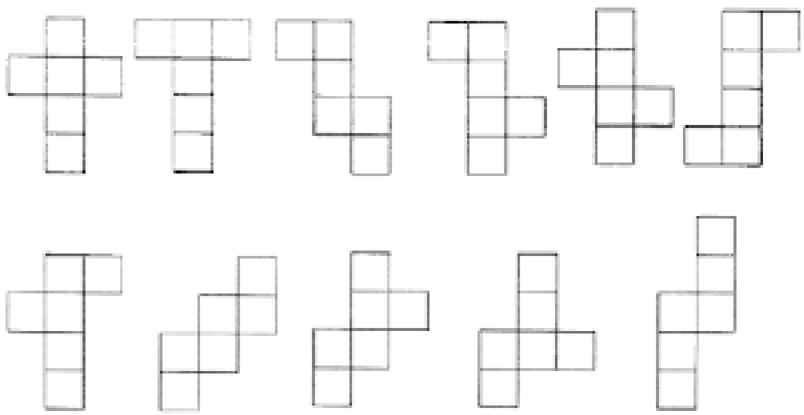


Figure 8 (The complete list of "nets" for a regular 1x1x1 cube. They arise from cutting the edges of the cube that form spanning trees of the cube.)

Consider the diagram shown in Figure 9.

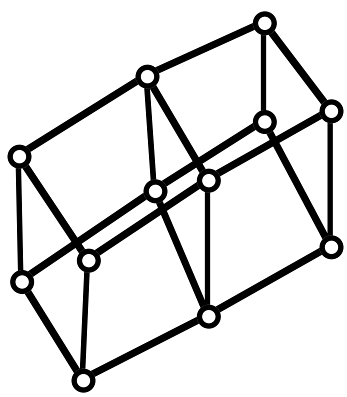


Figure 9 (A 1x1x2 box but when thought of as a convex polyhedron it has the same combinatorial type as the 3-dimensional cube - 8 vertices, 12 edges, and 6 faces. As shown it is not "strictly" convex because some of its faces are subdivided by "diagonals." However, there are interesting things to think about from considering the graph of this box as shown.)

This box can be considered as a 3-dimensional polyhedron, a drawing of a "polyhedron" in the plane, and a graph. As a graph one sees 12 vertices and 20 edges, but in Figure 9 edges intersect at points which are not vertices so this drawing does not allow one to think of the faces of the drawing. However, it can be drawn as a planar graph which is 3-connected but in this "incarnation" it is not the graph of a cube. It has 10 faces, all of which are 4-gons. As a polyhedron one might call this a 1x1x2 box assuming that the "drawing" in Figure 9 were "more exact" and that one is looking at two 1x1x1 cubes pasted together along one of the square faces of each cube. However, as a polyhedron this "object" in 3-space has a face, say the top face, which while it seems to have 6 sides is not truly a hexagonal face of the polyhedron. As a polyhedron, this is not the "convex hull" of its extreme points. The 1x1x2 box in Figure 9 can be folded from the assemblages of 4-gons, all squares in the top diagram, shown in Figure 10. On the bottom of Figure 10 we see a collection of polygons that can be folded to the box in Figure 9 and where one can see the four 1x2 rectangles clearly.

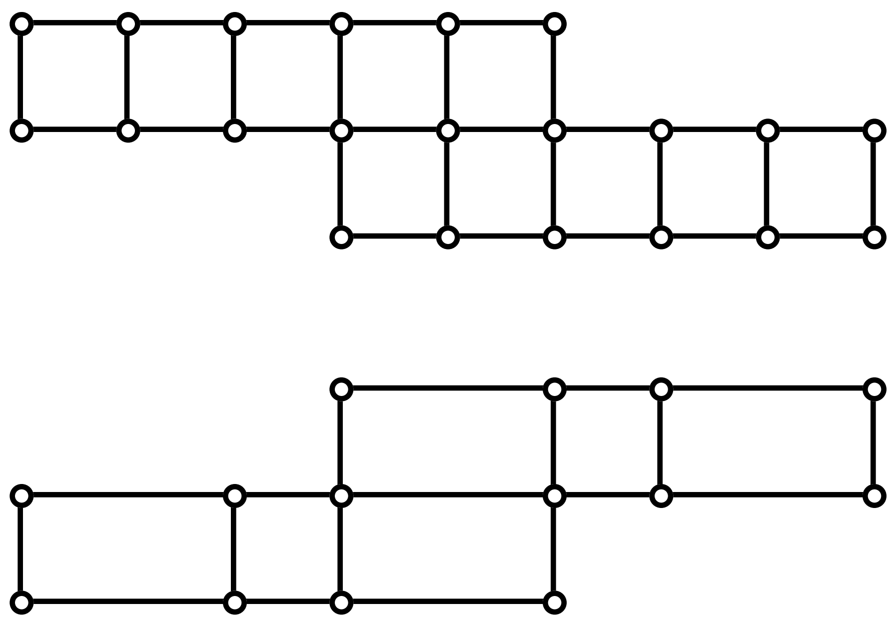


Figure 10 (Both of these "nets" will fold to a 1x1x2 box but only the second net makes clear which are the 1x2 rectangles that are faces of the solid.

The squares in Figure 11 can also be folded to a 1x1x2 box but the 1x2 rectangles of this box have to be "subdivided" when one cuts the box to make this "net" for the solid. Mathematicians have a special name for geometric objects that arise by putting together 1x1 squares edge to edge in the plane without holes (some authors allow holes). Such an assemblage is called a *polyomino*.

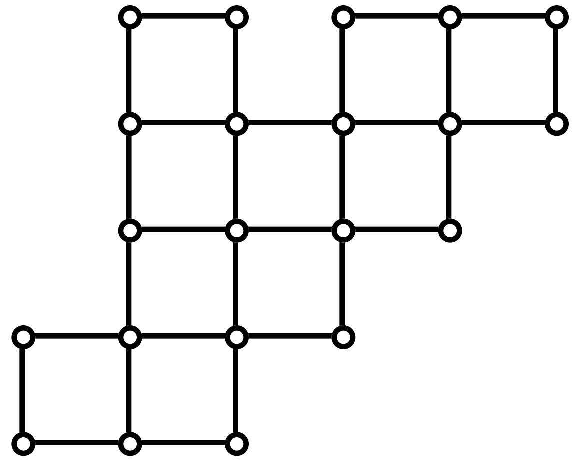


Figure 11 (A polyomino (plane collection of 1x1 squares jointed by edges) which will fold to a 1x1x2 box such as that in Figure 9.)

I hope the considerations above indicate why I encourage people who write about 3-dimensional convex polyhedra to use plane 3-connected graphs as part of their exposition.

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